Math 4150/6150, Optional Problems, Due date: 11/27/2018

1. Consider the square $S := [0,1] \times [0,1] \subset \mathbb{R}^2$. Find a decomposition $S = A \cup B$, such that $A \cap B = \emptyset$, and A, B are both connected, and $\{(0,0),(1,1)\} \subset A$, $\{(0,1),(1,0)\} \subset B$. (Recall that X being connected means it cannot be written as $X_1 \cup X_2$, where X_i are contained in U_i for $i = 1, 2$. Here, U_i are two disjoint open sets.) Prove your decomposition has this property.

2. In the course, we have **defined**

$$e^{i\theta} := cos\theta + isin\theta,$$

   where the right hand side works only as a short-hand notation. Historically, it is called the *Euler's formula*, which can be proved if we define e^x by its Taylor expansion. This justifies the short-hand notation.

   (i) Recall a definition of the exponential function is $e^x := \sum_{i=0}^{\infty} \frac{x^n}{n!}$ for all real x. Write down, in $\epsilon - \delta$ language, a statement that you think can help extend this definition of exponential function to complex value x. *You don't need to prove this statement.*

   (ii) Assume in step (1), you extend the definition of exponential function e^x such that x can be any complex number. Formally prove the Euler formula by using Taylor expansions of e^x, $sin\theta$ and $cos\theta$. (Plug in correct variables and manipulate the Taylor expansions).

   (iii) In your formal prove in (2), which step is not rigorous? Write down a statement which, if you could've proved it, can make your argument in (2) mathematically rigorous. *You don't need to prove this statement.*

3. Any complex function $f(z)$ can be interpreted as a complex-valued function with two real variables $f : \mathbb{R}^2 \to \mathbb{C}$ using the identification between $\mathbb{C}$ and $\mathbb{R}^2$.

   (i) Prove: one may use a change of variables $z = x + iy$ and $\bar{z} = x - iy$ to turn a complex valued function with variables x and y into a function with variables z and $\bar{z}$. Therefore, the complex function can be denoted as $g(z, \bar{z})$. Use an appropriate complex-valued matrix to describe this change of variales from (x, y) to $(z, \bar{z})$.

   (ii) Regard the change of coordinates from (x, y) to (s, t) as a composition of functions $x = x(s,t), y = y(s,t)$ (recall the example of polar coordinates in class). Then the chain rule says

$$\partial_s f = \partial_x f \partial_s x + \partial_y f \partial_s y$$
$$\partial_t f = \partial_x f \partial_t x + \partial_y f \partial_t y.$$

   Or, this can be written in a matrix form

$$\begin{pmatrix} \partial_s f \\ \partial_t f \end{pmatrix} = \frac{\partial(x,y)}{\partial(s,t)} \cdot \begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix}.$$

   Use the above formula and (i) to prove that, $g(z, \bar{z})$ is holomorphic if and only if $\frac{\partial}{\partial \bar{z}} g(z, \bar{z}) = 0$.

   (iii) As an application, use (2) to prove $|z|^2$, $Re(z)$ and $Im(z)$ are not holomorphic.

4. Given a circle in $\mathbb{C}$ as $\{|z-a|=R\}$. Find all Mobius transforms that preserve this circle. Prove that the family of Mobius transforms you found exhausts all possibilities.

5. Is it possible to define a logarithm $\widetilde{Log}(z)$ over $G := \mathbb{C}\backslash\{z = x+yi : y = x^2, x \geq 0\}$, such that

   (i) $\widetilde{Log}(z)$ is continuous on G, and

   (ii) $exp(\widetilde{Log}(z)) = z$?

   If you think it is possible, give the expression and prove it satisfies the above two requirements; if not, prove it.

6. (1) Give an example of linear transformation $J : \mathbb{R}^n \to \mathbb{R}^n$, such that $J^2 = -I$ for some n. Show that n must be even to admit such a linear transformation.

   (2) Given a real vector space $V = \mathbb{R}^{2k}$ equipped with a J as in (1). Consider its *complexification* $V \otimes \mathbb{C} := \{v = (z_1, z_2, \cdots, z_{2k}) : z_i \in \mathbb{C}\}$. This forms a vector space over the complex number field, and you should be able to verify all axioms of a vector space (but you don't need to turn in this verification in the homework).

   Show that J extends to a $\mathbb{C}$-linear transformation on $V \otimes \mathbb{C}$, and it has two eigenvalues $\{i, -i\}$. Let $V^{1,0}$ and $V^{0,1}$ denote the eigenspace of i and $-i$, respectively. Prove $V^{1,0} \cong V^{0,1}$ and write down the isomorphism explicitly; and prove that $V \otimes \mathbb{C} \cong V^{1,0} \oplus V^{0,1}$.

7. Given a pair of real-valued function $f(x)$ and $g(x)$, we can define a *Riemann-Stieltjes integral*, denoted as

$$\int_a^b f(x)dg(x).$$

   Let $P := \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of interval, and Δ_P be the length of the longest sub-interval, that is, $\Delta_P := \max_i(x_{i+1} - x_i)$. Denote the sum

$$S(P, f, g) = \sum_{i=0}^{n-1} f(c_i)(g(x_{i+1}) - g(x_i)),$$

   then suppose $\lim_{\Delta_P \to 0} S(P, f, g) = A$ exists, then the limit A is the Riemann-Stieltjes integral.

   (i) Write down in ϵ-δ Language the statement $\lim_{\Delta_P \to 0} S(P, f, g) = A$.

   (ii) The Dirichlet function is defined as

$$D(x) = \begin{cases} 0, & \text{if } x \text{ is rational,} \\ 1, & \text{if } x \text{ is irrational,} \end{cases} \tag{0.1}$$

   Find all functions $F(x)$ such that $\int_0^1 F(x)dD(x)$ exists. Find all functions $G(x)$ such that $\int_0^1 D(x)dG(x)$ exists.

   (iii) Assume in the definition of Riemann-Stieltjes integral that $g(x)$ is smooth. Explain how to regard it as a line integral in complex analysis.

8. (1) Prove a mean value property for harmonic function $\frac{1}{\pi}\iint_{D[(x_0,y_0),r]} u(x,y)dxdy = u(x_0,y_0)$. Note that this is slightly different from what is on the textbook, where the integration is a contour integral. You might want to use polar coordinate.

   (2) Use Green's theorem and the mean value property of harmonic functions to prove Cauchy's integral formula using multi-variable calculus.

$$\int_{|z|=1} \frac{f(z)dz}{z-a} = f(0).$$

9. In this problem, we use notation from 3. Let $|w| < 1$, $f(z) = u(x,u) + iv(x,y)$ be a complex-valued (necessarily holomorphic) function, and assume u, v are both differentiable. Then

$$f(w) = \frac{1}{2\pi i}\int_{C[0,1]} \frac{f(z)}{z-w}dz - \frac{1}{2\pi i}\iint_{D[0,1]} \frac{\partial f(z)}{\partial \bar{z}} \cdot \frac{d\bar{z}\wedge dz}{z-w}.$$

10. In this problem, we use notation from 3.

    Consider a function with two complex variables $f : \mathbb{C}^2 \to \mathbb{C}$, and the two variables are denoted z_1, z_2. We call f holomorphic at $(z_1, z_2) = (a, b)$, if both one-variable complex functions $g_1(z) := f(z, b)$ and $g_2(z) := f(a, z)$ are holomorphic.

    Denote $g_1'(a)$ and $g_2'(b)$ as $\frac{\partial f}{\partial z_1}(a,b)$ and $\frac{\partial f}{\partial z_2}(a,b)$, respectively.

    (i) Give a reasonable definition for $\dfrac{\partial f}{\partial \bar{z}_1}(a,b)$. Write down the Cauchy Riemann equations for $f(z_1, z_2)$ to be holomorphic. Show that f is holomorphic if and only if

$$\frac{\partial f}{\partial \bar{z}_1} = \frac{\partial f}{\partial \bar{z}_2} = 0$$

    (ii) Suppose $f(z_1, z_2) = F(z_1, z_2)/G(z_2)$ where both $F(z_1, z_2)$ and $G(z_1, z_2)$ are both holomorphic on $\mathbb{C}^2$, and , that is, the first coordinate plane. One may compute the integral

$$g(z) := \int_{|w|=1} f(z,w)dw$$

    Prove $g(z)$ is holomorphic.