2.6 Exercises

1. For semivariogram models #2, 4, 5, 6, 7, and 8 in Subsection 2.1.3,
   (a) identify the nugget, sill, and range (or effective range) for each;
   (b) find the covariance function $C(t)$ corresponding to each $\gamma(t)$, provided it exists.

2. Prove that for Gaussian processes, strong stationarity is equivalent to weak stationarity.

3. Consider the *triangular* (or "tent") covariance function,

$$C(\|h\|) = \begin{cases} \sigma^2(1 - \|h\|/\delta) & \text{if } \|h\| \leq \delta,\ \sigma^2 > 0,\ \delta > 0, \\ 0 & \text{if } \|h\| > \delta \end{cases} .$$

It is valid in one dimension. (The reader can verify that it is the characteristic function of the density function $f(x)$ proportional to $[1 - \cos(\delta x)]/\delta x^2$.) Now in two dimensions, consider a 6×8 grid with locations $\mathbf{s}_{jk} = (j\delta/\sqrt{2}, k\delta/\sqrt{2})$, $j = 1, \ldots, 6$, $k = 1, \ldots, 8$. Assign a_{jk} to $\mathbf{s}_{jk}$ such that $a_{jk} = 1$ if $j+k$ is even, $a_{jk} = -1$ if $j+k$ is odd. Show that $Var[\Sigma a_{jk} Y(\mathbf{s}_{jk})] < 0$, and hence that the triangular covariance function is *invalid* in two dimensions.

4. The *turning bands method* (Christakos, 1984; Stein, 1999a) is a technique for creating stationary covariance functions on $\Re^r$. Let $\mathbf{u}$ be a random unit vector on $\Re^r$ (by random we mean that the coordinate vector that defines $\mathbf{u}$ is randomly chosen on the surface of the unit sphere in $\Re^r$). Let $c(\cdot)$ be a valid stationary covariance function on $\Re^1$, and let $W(t)$ be a mean 0 process on $\Re^1$ having $c(\cdot)$ as its covariance function. Then for any location $\mathbf{s} \in \Re^r$, define

$$Y(\mathbf{s}) = W(\mathbf{s}^T\mathbf{u}) .$$

Note that we can think of the process either conditionally given $\mathbf{u}$, or marginally by integrating with respect to the uniform distribution for $\mathbf{u}$. Note also that $Y(\mathbf{s})$ has the possibly undesirable property that it is constant on planes (i.e., on $\mathbf{s}^T\mathbf{u} = k$).

   (a) If W is a Gaussian process, show that, given $\mathbf{u}$, $Y(\mathbf{s})$ is also a Gaussian process and is stationary.
   (b) Show that marginally $Y(\mathbf{s})$ is *not* a Gaussian process, but is isotropic. [*Hint:* Show that $Cov(Y(\mathbf{s}), Y(\mathbf{s}')) = E_{\mathbf{u}} c((\mathbf{s} - \mathbf{s}')^T\mathbf{u})$.]

5.(a) Based on (2.10), show that $c_{12}(\mathbf{h})$ is a valid correlation function; i.e., that G is a bounded, positive, symmetric about 0 measure on $\Re^2$.
   (b) Show further that if c_1 and c_2 are isotropic, then c_{12} is.

6.(a) What is the issue with regard to specifying $\widehat{c}(0)$ in the covariance function estimate (2.15)?