A splitting criterion for an isolated singularity at x = 0 in a family of even hypersurfaces

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Abstract

Assume given a family of even local analytic hypersurfaces, whose central fiber has an isolated singularity at x = 0 which is not an ordinary double point. We prove that if the family is sufficiently general, for instance if the general fiber is smooth and the general singular fiber has only ordinary double points, then the singularity at x = 0 "splits in codimension one", i.e. the local discriminant divisor has an irreducible component, over which a general fiber has more than one singularity specializing to the original one. As a corollary, we deduce the result [G-SM2, Prop.8] that on a principally polarized abelian variety (A, “theta”) with dim(A) = g ≥ 4, a singularity of even multiplicity on “theta”, isolated or not, at a point of order two and not an ordinary double point, must be a limit of two distinct ordinary double points (x,-x} on nearby theta divisors.

Introduction

The study of singularities on theta divisors of principally polarized abelian varieties (ppav's) has long been of interest, for instance in connection with the problem of recognizing which ppav's are Jacobians of curves [A-M]. For example let A(g) be the moduli space of g dimensional ppav's (A, “theta”), N(k) the subvariety of A(g) where "theta" has a singular locus of dimension ≥ k, and “Thetanull” the subvariety of A(g) where "theta" contains a point of order two with even multiplicity (a "vanishing even thetanull" or "thetanull" for short). Then for g ≥ 4, the subvariety J(g) consisting of (the closure) of Jacobians is an irreducible component of N(g-4) not contained in “Thetanull”, and the subvariety of hyperelliptic Jacobians is a component of N(g-3). These descriptions characterize Jacobians and hyperelliptic Jacobians for g ≤ 5 [B], but it is unknown whether this holds also for g ≥ 6.

When g ≥ 4 the subvariety N(0) has one component other than Thetanull, denoted N'(0), and one can show (see below) that N(1), hence for g ≥ 5 also N(g-4), is contained in N'(0). Thus the questions above lead one to study (N'(0)meetThetanull), as in [D, G-SM1,2]. By definition (A,"theta") belongs to Thetanull if and only if "theta" has a (vanishing even) thetanull, and in fact (A,"theta") belongs to N'(0) if and only if "theta" has a singularity which is in the closure of singularities which are not thetanulls. In particular if "theta" has a thetanull and also a singularity which is not a thetanull then (A,"theta") lies on (N'(0)meetThetanull), but there are also some (A,"theta") in "Thetanull" with only thetanulls as singularities, and which also belong to N'(0). The question of how to recognize them is answered by the theorem of [G-SM2, Prop.8] (whose proof implies) that a thetanull is a limit of singularities which are not thetanulls, if and only if it is not an ordinary double point.

The proof in [G-SM2] applies only to singularities of theta divisors, using the heat equation to relate the analytic conditions defining these two properties for singularities of theta functions. They note that a singular point p of a theta divisor lying in (N'(0)meetThetanull) must be singular on the critical locus of the universal family, and use the heat equation to relate singularity of the critical locus at p to the dropping of the rank of the tangent quadric to "theta" at p.

The goal of this paper is to prove a general deformation theoretic statement which applies to all even local analytic hypersurfaces and implies their result as a consequence. We show that in any sufficiently general deformation of an even analytic function having an isolated singularity at x = 0 of positive corank (Defn. I.3 below), there is a pair of distinct conjugate singularities {x,-x} converging to the singularity at x = 0. The argument is inductive, building on the one in [S-V], which assumed the singularity has corank exactly one. A density argument as in [G-SM2, lemma 9], using the theorem [C-vdG, Th.8.6] that the locus N(1) of theta divisors having positive dimensional singular locus has codimension ≥ 3, implies the result also in the case of a theta function with non isolated singularities. We hope the new perspective may lead to further insight into some open questions on theta divisors which we discuss at the end, and perhaps have more general applications.

In this paper we work over the complex numbers.

I. Definitions and background

I.1. Definition: By a family of local analytic hypersurfaces we mean one defined by an analytic function F(x;s) defined on some nbhd UxV of (0;0) in C^n x C^r, together with the projection map UxV-->V taking (x;s) to s. We assume that for all s in V, {x: F(x;s) = 0} is a non empty hypersurface in U, called the fiber over s. This means, for all s in V, the function F(x;s) on U is neither identically zero nor a unit.

The fiber over s = 0 is called the central fiber. We assume F(0;0) = 0, i.e. that (0;0) is on the central fiber. We may also call the family simply "F", and we may assume after shrinking them that both U and V are connected.

We say F is "even in x", or just "even" if F(x;s) = F(-x;s) for all (x;s) in UxV (and U is invariant under the map x--> -x). A point of the fiber over s is singular on that fiber if F and all x - partials of F vanish at the point. An isolated singular point of a fiber is one such that in some neighborhood, no other point of that fiber is singular.

I.2. Definition: If F(x;s) defines a family of local analytic hypersurfaces such that (0;0) is an isolated singular point of the central fiber, we say this singularity "splits" (locally in the family), if in every neighborhood of (0;0) in UxV some fiber of F has more than one singularity, equivalently if there exist values of s approaching 0 in V such that over each such s there are at least two distinct singularities $p(s) \neq q(s)$, and such that p(s)-->0 and q(s)-->0 in U as s-->0 in V. A fiber is smooth or non-singular if it contains no singularities. We say (0;0) "smoothes" (locally in the family) if there is a basis of neighborhoods of (0;0) each containing some non-singular fibers of F.

I.3. Definition: The "rank" of a hypersurface singularity is the rank of the homogeneous quadratic term of its Taylor series at the point. Thus, for a hypersurface singularity in C^n, the rank may take values between 0 and n; the rank is n if and only if the point is an ordinary double point (odp), and the rank is 0 if and only if the singular point has multiplicity greater than 2. The corank of a singularity in C^n is defined as (n-rank). Thus a singularity has positive corank if and only if it is not an odp.

We want a criterion for an isolated even hypersurface singularity to split in a given family. Since an odp never splits [K-S], it is necessary that a singularity which splits have positive corank. We shall prove that if the function F is even in x, and the family is otherwise sufficiently general (in a sense to be defined), then an isolated singularity at (0;0) splits if and only if the corank is positive. In fact we will prove more, that an isolated singularity at (0;0) in a sufficiently general family of even hypersurfaces "splits in codimension one" (to be defined later) if and only if it is not an odp. A general family, roughly speaking, is one whose general fiber is non-singular, and whose general singular fiber has only odp's. The argument will show that a slightly weaker definition suffices, as will be explained after we define "splitting components" in the "essential discriminant locus" of a smoothing family with isolated singularity in the central fiber.

I.4. Definition: Given a family of hypersurfaces in UxV with singularity at (0;0), the set S of all singular points of all fibers of the family is the "critical locus" of the family. Then S is a closed analytic subset of UxV containing (0;0). If U is open in C^n, S is defined by n+1 equations (F and its x- partials) hence S has codimension at most n+1 in UxV.

If the family F has an isolated singularity at (0;0) in the central fiber, we may shrink U until (0;0) is the only singularity in the central fiber. If we shrink the neighborhood UxV of (0;0) further (to have compact closure in the domain of F), by the argument in [lemma, p.44, M] there is a smaller open nbhd of 0 in V, over which the projection S-->V is proper. Then the restriction of this projection over any smaller subset of V, open or not, remains proper. Since the fibers are then compact analytic subsets of C^n, all fibers of the map S-->V are then finite [G-R, p.106].

I.5. Definition: If F is a family of analytic hypersurfaces in UxV with critical locus S such that the projection S-->V is proper, then the image of S in V, is a closed analytic set N called the "discriminant locus" of the family. Then s lies in N if and only if the fiber of F over s has a singular point. (In this generality, N could be empty or equal to all of V.)

We want to add hypotheses guaranteeing that N has pure codimension one in V.
I.6. Definition: We shall say F satisfies condition (*) if it defines an analytic family of hypersurfaces in UxV such that:
i) the central fiber has a singularity at (0;0) and nowhere else;
ii) the singularity (0;0) smoothes in the family;
iii) the projection S-->V is proper with finite fibers.

I.7. Remark: For every family F having an isolated singularity at (0;0) which smoothes, we can always shrink UxV as indicated above until the restricted F satisfies (*). Once this is done, (*) is preserved under further shrinking of the open neighborhood V of 0.

Since each irreducible component Z of S has codimension $\leq$ dim(U)+1 in UxV, it follows that dim(Z) $\geq$ dim(V)-1. If (*) holds, and W is the image of Z in V, then dim(W) = dim(Z), and the smoothing hypothesis implies S-->V is not surjective. Hence we have dim(V) – 1 $\leq$ dim(Z) = dim(W) $\leq$ dim(V)-1. Hence (*) implies all components of S and of N have dimension dim(V)-1.

I.8. Definition: We call a component of the critical locus S "essential" if it contains (0;0), and call its image in V an essential component of the discriminant locus.

I.9.Remark: Assuming (*), a component of the discriminant locus N is essential if and only if it contains s = 0. In fact, the inverse image under the finite map S-->N of each essential component W of N must have the same dimension as W, hence must be a union of irreducible components of S, each mapping onto W since W is irreducible. Hence if W is essential in N, each component of the inverse image of W is essential in S, by properness and the uniqueness of the singularity of the central fiber.

By shrinking V until N has only a finite number of components in V, and then removing those components of N that do not contain 0, we can arrange that N has only a finite number of components, all of which contain 0, i.e. all of which are essential. By the previous arguments, then all components of S are also essential.

Thus beginning from any analytic family of hypersurfaces in UxV whose central fiber has an isolated singularity at (0;0) which smoothes locally, we can always shrink U until x = 0 is the only singularity of the central fiber, and then shrink V until (*) holds and all components of both N and S are essential.

I.10. Definition: Assuming (*) holds, an essential component W of N is a "splitting" component for (0;0) if the degree of S-->N over W is greater than one, and "non - splitting" if the degree is one. Thus W in N is a splitting component if either the inverse image of W in S has more than one component, or for some component Z of the inverse image of W in S, the map Z-->W has degree > 1. If N has an essential splitting component, we say the singularity (0;0) splits in codimension one.

I.11. Definition: A family F of local hypersurfaces with an isolated singularity at (0,0) is called "sufficiently general" if:
1) the singularity (0,0) smoothes in the family, and
2) there exists a (*) neighborhood UxV of (0,0) such that over a general point of every essential, non-splitting component of the discriminant locus N in V, the only singularity of the fiber is one ordinary double point.

I.12. Remark: A family of local hypersurfaces with an isolated singularity at (0,0) is sufficiently general if there is a (*) neighborhood of (0;0) in which the general fiber is smooth, and over every component of the discriminant locus, the general fiber has only odp's.

II. Statement of the main theorem and application to theta divisors

The following theorem gives a general splitting criterion in the isolated case.

II.1. Theorem: Assume F(x;s) is an analytic family of local, even, hypersurfaces as in definition I.1, whose central fiber has an isolated singularity at (0;0). If F is "sufficiently general" in the sense of definition I.11, and the singularity at x = 0 of the central fiber is not an odp, then there is a (*) neighborhood UxV of (0;0) for which the discriminant locus N has an essential splitting component.

II.2. Cor: In a sufficiently general family of local even analytic hypersurfaces as in definition I.1, an isolated singularity at (0;0) splits locally if and only if it splits in codimension one, if and only if it is not an odp.
proof of Cor. II.2: By [K-S] the versal family for an odp has no splitting fibers at all. Since every deformation of an odp on a local hypersurface is a pull back of this versal family, an odp can never split, hence cannot split in codimension one. The converse follows from the theorem. **QED.**

Recall A(g) is the moduli variety of g dimensional ppav's, and N(0) = N'(0) union "Thetanull" is the discriminant locus of the universal family of theta divisors, the sublocus of those (A,"theta") in A(g) such that "theta" is singular. "Thetanull" is the locus of (A,"theta") such that "theta" has a thetanull, i.e. a point of positive even multiplicity occurring at a point of order two for A, and N'(0) is the other component, whose general element has singularities on "theta" but no thetanulls. It is known that for (A,"theta") general on "Thetanull", the only singularity of "theta" is a single odp at a point of order two [D2; G-SM2, Remark 6], and for (A,"theta") general on N'(0) the only singularities are two distinct odp's {x,-x} not at points of order two.

To study the components "Thetanull" and N'(0) of the discriminant locus in A(g) for the "universal family" of theta divisors, we must examine the critical locus of that family. Choose a finite level covering of A(g) over which a family of ppav's (A, "theta") exists, in particular a family of theta divisors hence also the critical locus S comprised of the union of the singular loci of "theta" for all (A,"theta") in the family. By [C-vdG, Cor.8.10] every component of S has the same dimension, and S decomposes naturally into two subsets, Snull and S' as in [D1, G-SM2] where Snull is the set of thetanulls and S' is the union of the other components of S. Since the condition of having a vanishing even thetanull is closed, the set Snull of thetanulls is a union of components of S. Since the only singularity over a general point of "Thetanull" is just one thetanull, all components of Snull dominate "Thetanull" and no components of S' do so. By [C-vdG, Cor.8.10], S' decomposes further into S'' and scriptE, where S'' is the union of those components of S that dominate N'(0), and script(E) is the union of those components of S whose general fiber consists of the singular locus of "theta"(ExB), where E is an elliptic curve and B a general ppav of dimension g-1. Thus S = Snull union S'' union script(E), where Snull is the union of those components of S which dominate "Thetanull", S'' is the union of those components that dominate N'(0), and script(E) is the union of those "exceptional" components of the critical locus S whose image in A(g) is a lower dimensional "embedded" subvariety of the discriminant locus N(0). By [D1,2; G-SM2] the general point of a component of both S'' and Snull is an odp on its respective theta divisor, (but not on script(E)). Define S^(g-1)null to be the subset of Snull consisting of thetanulls of positive corank, and "Theta^(g-1)null" to be the image of S^(g-1)null in

“Thetanull”, i.e. (A, “theta”) in A(g) belongs to “Theta^(g-1)null” if and only if “theta” contains a thetanull of positive corank.

II.3. Cor [G-SM2]: With notation as above, in the universal critical locus of a family of theta divisors over (a level cover of) A(g) with g ≥ 4, we have S^(g-1)null = S’’ meet Snull = S’ meet Snull. I.e. if a thetanull p on a ppav (A, “theta”) of dimension g ≥ 4 has positive corank, then p is a limit of distinct odp’s {x,-x} lying on nearby theta divisors. Thus in A(g) the set “Theta^(g-1)null” belongs to the intersection “Thetanull” meet N’(0), and hence (by dimension count) is a component of that intersection.

Proof of Cor. II.3: By [K-S] an odp cannot be a limit of distinct conjugate singularities {x,-x}, so (S’’ meet Snull) is contained in S^(g-1)null. Since script(E)meetSnull is contained in S^(g-1)null, it only remains to show that S^(g-1)null is contained in S’’meetSnull. First suppose p is in S^(g-1)null and an isolated singularity on “theta”. Riemann’s theta function is even (locally near a point of order two of even multiplicity on “theta”), a general ppav in A(g) has smooth theta divisor [A-M], and over a general point of every component of the discriminant locus the “theta” has only odp’s [D, G-SM2]. It follows from the theorem that p splits in codimension one, hence is a limit of a pair of distinct conjugate odp’s on S’’. Thus for g ≥ 4, points of S^(g-1)null which are isolated singularities lie on S’’ meet Snull.

Now suppose the point p is a non isolated singularity of theta. By [C-vdG, (A-M paper) Cor.8.10], when g ≥ 4 the locus N(1) of ppav’s having a non isolated singularity on theta has codimension ≥ 3 in A(g). On the other hand, as observed in [G-SM2, the locus of ppav’s having a singularity of positive corank at some “even” point of order two, is of pure codimension 2 in A(g), being cut out on Thetanull by a single equation, the vanishing of the Hessian of second partials of the variables on the abelian variety A. Since the space of points of order two lies finitely over A(g), the same equation shows that S^(g-1)null has codimension two in all even points of order two. Since a finite map preserves dimension, those vanishing even thetanulls which are non isolated singularities of theta, i.e. those lying over N(1), have codimension at least 3 in the space of all even points of order two. Thus those points of S^(g-1)null which are isolated singularities of “theta” are dense in S^(g-1)null. Thus again p lies on S’’meetSnull, for all p in S^(g-1)null, even if p is a non isolated singularity of “theta”. **QED.**

II.4. Cor: In particular a four dimensional ppav with a vanishing even theta null p as an isolated singularity on its theta divisor, is a Jacobian of a smooth curve of genus 4 if and only if the corank at p is positive, if and only if p is a rank 3 double point, (Farkas’ conjecture [F], as proved in [G-SM1], cf.also [SV1] for the rank 3 case).

II.5. Remark: Since the critical locus S in the “universal family” over A(g), is separated into the sets, Snull, S’’ and script(E), the intersection “Thetanull”meetN’(0) can also be decomposed naturally. Let R = the image in A(g) of S^(g-1)null = the image of S’’meetSnull = those ppav’s having theta nulls which are limits of singularities which are not thetanulls. By dimension count, as in [G-SM2] this is a union of components of “Thetanull”meetN’(0). Let D denote the union of the other components of “Thetanull”meetN’(0). Thus D consists of components of “Thetanull”meetN’(0) at a general point of which “theta” has some thetanulls, all of which are odp’s, and “theta” also has some other singularities which are not thetanulls.

II.6. Cor. The component N’(0) consists of those ppavs in A(g) on which "theta" has a singularity which is in the closure of singularities which are not thetanulls. In particular, any (A,"theta") such that "theta" has a singularity which is not a thetanull is in N’(0), hence N(1) (and N(g-4), for $g \geq 5$) lies in N’(0). In particular, if (A,"theta") does not lie on N'(0), then there are no singularities on “theta” except thetanulls, and all thetanulls that exist on "theta" are odp's.
proof: Since all singularities which are not thetanulls belong to S’, we only have to show the image of S’ lies in N’(0). This is clear for S’’ since all components of S’’ dominate N’(0). Since a product ppav always has a thetanull of positive corank, (take a point on ExB of form (p,q) where p,q are points of order two and odd multiplicity on “theta”(E), “theta”(B) respectively), the image of the critical component script(E) of S lies entirely in the component R of “Thetanull”meetN’(0). **QED.**

III. Remarks and examples in the general non isolated case:

By our definition of splitting, a non isolated singularity at x = 0 on an even hypersurface always splits, since it is a limit of non zero conjugate singularities on the central fiber itself. Codimension one splitting in the base space is another matter. For that, the property that N(1) has codimension at least three in the base is essential. For example, the even affine hypersurfaces {X^2 + Y^2 +s Z^2 + t = 0} over the (s,t) plane are smooth for $t \neq 0$, and singular for t = 0. If t = 0 but $s \neq 0$, there is a single odp at (0,0,0). For s = t = 0, there is a curve of singularities passing through (0,0,0), so the locus N(1) is the origin (0,0) of codimension two in the (s,t) plane. The non isolated double point at the origin has corank one and there is no codimension one splitting component of the discriminant locus.

This resembles the situation in A(3) where the discriminant locus of hyperelliptic Jacobians is irreducible of codimension one, and contains the codimension two subvariety of products. The singular loci on theta divisors of products are all curves, and on a hyperelliptic Jacobian “theta” has a single odp, so there is no codimension one splitting of the non isolated vanishing even thetanulls.

If we alter the family slightly to the situation for a genus 4 Jacobian with a vanishing even thetanull, we get X^2 + Y^2 +Z^2 + W^4 +sW^2 + t = 0. Again the discriminant locus contains the axis t = 0, but now when t = s = 0, the rank drops without the dimension of the singular locus going up, since (0,0,0,0) is an isolated singularity of X^2 + Y^2 +Z^2 + W^4 = 0. This forces the existence of a splitting component, namely the curve {s^2 - 4t = 0}, on which a general hypersurface has two odp’s. The variety N(1) here is empty, i.e. of codimension three in the (s,t) plane.

In the example X^2 + Y^2 +Z^2 + rW^4 +sW^2 + t = 0, the locus N(1) is the origin of (r,s,t) space, again of codimension three. The point (X,Y,Z,W) = (0,0,0,0) is a non isolated double point on X^2 + Y^2 +Z^2 = 0, and splits in codimension one along the surface {s^2 – 4rt = 0} in the discriminant divisor {t(s^2-4rt) = 0}.

When all components of the critical locus have the expected dimension, as occurs in A(g), all that is needed for N(1) to have codimension three is that the general singular locus is never one dimensional on any component of the critical locus. Since the general singularity of “theta” is isolated when (A, “theta”) is indecomposable [C-vdG], and of dimension g-2 when (A,theta) is a product, N(1) has codimension > 2 in A(g) for $g \geq 4$.

IV. Proof of the main Theorem II.1:

Since F is even, we may assume after shrinking that U is invariant for $x \to (-x)$. We want to show that if (0;0) is not an odp then it splits in codimension one, i.e. there is a (*) neighborhood on which there is an essential splitting component of the discriminant.

Step one) Assume the singularity (0;0) on the central fiber is isolated of corank exactly one. Then by the preparation lemma in [S-V] there exist analytic coordinates (z(x;s);s) near (0;0), with $z(-x;s) = -z(x;s)$, such that the family is defined by $F(z;s) = z_1^2+\dots+z_{n-1}^2 + h(z_n^2;s)$, where $h(y;s) = y^k + a_{k-1}(s)\, y^{k-1} + \dots + a_0(s)$, is a polynomial in y of degree k with analytic coefficients $a_j(s)$ with all $a_j(0) = 0$, and $k \geq 2$.

Choose a (*) neighborhood UxV of (0,0) in these coordinates where this representation holds. Let $a: V \to P$ be the analytic map to the space $P \approx \mathbb{C}^k$ of monic polynomials of degree k in y, defined by $a(s) = h(y;s)$. Let D1 be the divisor in P of polynomials h(y) having a multiple root y and D0 the divisor of polynomials h(y) with zero constant term. For s in V, the function F(z;s) has a singularity at z = 0 if and only if $F(0;s) = 0$, if and only if $a_0(s) = 0$, if and only if a(s) lies on D0.

Since $a(0) = h(y;0) = y^k$ lies on both D1 and D0, and has all its singular points at y=0 in U, and the polynomials h(y;s) are monic, we may choose V so small that for all s in V, all singularities y of the polynomial h(y;s) not only lie in U but have all their square roots in U. Then every singularity y of h(y,s) gives rise to (not necessarily distinct) singularities {z,-z} in U of $F(z;s) = z_1^2+\dots+z_{n-1}^2 + h(z^2;s)$, where $z^2 = y$.

Thus the divisors D1 and D0 in P pull back under the map a, to non empty divisors D1*, D0* in the discriminant locus in V of the family F, both of which contain s = 0. Moreover, if $a(s) = h(y;s)$ lies in D1 but not D0, then h(y;s) has a singularity at some non zero value of y, and F(z;s) has as singularities both square roots of that non zero value, in particular F(z;s) has then at least two singularities. Thus if $a(s) = h(y;s)$ lies in D1, and yet F(z;s) has only one singularity in z, then that singularity occurs at z= 0, so that the singularity of h(y) also occurs at y = 0.

Now let W be an irreducible component of the divisor D1* containing s = 0. We claim W is a splitting component for (0;0). If not, then there is only one singularity over a general point s of W, and by the previous remarks $F(z;s) = z_1^2+\dots+z_{n-1}^2 + h(z^2;s)$, where its unique singularity is at z = 0, and h(y;s) also has a unique singularity, which occurs at y = 0. Then h(y) is divisible by y^2, hence $h(z^2;s)$ is divisible by z^4, so the singularity at z = 0, of $F(z;s) = z_1^2+\dots+z_{n-1}^2 + h(z^2;s)$ is not ordinary.

Since our family is assumed sufficiently general, this contradicts the hypothesis that over a general point of a non splitting component W the fiber has as singularity only one odp. Thus every irreducible component W of D1* which contains (0;0), i.e. every essential component of D1* in N, is a splitting component for (0;0). Thus (0;0) splits in codimension one in a sufficiently general family F. **QED for case 1).**

Step two) Now assume F is even and sufficiently general and satisfies (*) in UxV and that the isolated singularity of F(x;0) at x = 0 has corank greater than one, i.e. rank $k < n-1$. We wish to prove there exist essential splitting components for the discriminant locus, so we may assume there are none, i.e. that all components of N which contain 0, are non splitting, hence that over a general point of every component of N the only singularity is one odp. By shrinking V further we may assume as in Remark 9 that all components of both N and D are essential, hence the degree of S-->N is one over every component of N. We will obtain a contradiction by enlarging our family until it satisfies the hypotheses of step 1, producing a splitting component in the larger family, and proving that its intersection with the original family violates the given hypotheses.

We will enlarge V, to a product of V with another space parametrizing quadratic homogeneous polynomials. By adding enough quadratic polynomials Q(x) to the functions F(x;s), we obtain a new enlarged family of even functions defined on the same domain U in C^n, but parametrized by an open set V' in a higher dimensional space, and such that arbitrarily near s' = 0, there are now modified functions F'(x;s') which have (0;0) as an isolated singular point of corank one. The idea is to mimic the use of a general even "unfolding" of the original singularity, but in a minimal way, without invoking any general deformation theory.

For instance it suffices to add all homogeneous quadratic polynomials in x, or we could diagonalize the homogeneous quadratic term of F(x;0) as $x1^2+...+xk^2$, and just add to every F(x;s) all scalar multiples of $Q(x) = x(k+1)^2 +......+ x(n-1)^2$. Then over the point s' = (s, s(r+1)) in Vx(complex field) = V' we have the function F'(x;s') = F(x;s) + s(r+1)Q(x). We choose this last method, so that the original base space V is a smooth divisor in the enlarged one V' = Vx(complex field), which simplifies understanding the dimension of sets obtained by intersecting subsets of V' with V.

We must check that the enlarged family F' over V' is still sufficiently general. At least the central fiber has not changed so (0;0) is still the only singularity in it, and there are still (the same) smooth fibers in every nbhd of (0;0), so (0;0) smoothes in the new family. We have added only homogeneous quadratic polynomials to even functions, so the functions remain even.

If we denote by C the new critical locus in UxV' and by D the new discriminant locus in V', we can shrink V' until the projection from C-->D is proper with finite fibers as before, i.e. until (*) holds. By Remark I.9 we can also shrink until all components of D are essential. So it remains to check property 2) of "sufficiently general" in definition 11. So let Y' be any non splitting component of D; then Y' contains 0 and the degree of C-->D over Y' is one. We want the general fiber over Y' to be one odp. If we intersect Y' with the original V, we get a divisor in V over which all fibers are singular, hence all components of the intersection Y'meetV are in N. Since Y' contains 0, so does Y'meetV. Let Y be a component of Y'meetV that contains 0 (in fact every component does so by our choices).

Since we have assumed all components of N which contain 0 are non splitting, this is true for Y. Thus over a general point of Y the fiber has exactly one singularity, an odp. Moreover, the singularity over a general point of Y' is a small deformation of this odp, hence also one odp. I.e. once we find a single point of Y' over which the singular locus is just one odp, then that also holds for the general point of Y'. Since we have shown that over every essential non-splitting component of D the singularity is one odp, the new family F' is sufficiently general.

Moreover, since every component of D contains 0, in fact we have shown that over every component of D where the degree of C-->D is one, the general singularity is just one odp. This implies that the family also restricts locally to a sufficiently general deformation of every singularity in C occurring at x = 0 over any point s' in V'. We will exploit this fact next, by arguing that s' = 0 is a limit of points s' where step 1) applies.

Now, let s' be a point of V' such that x = 0 is an isolated singularity of corank 1 in the fiber over s'. Such points s' occur in every neighborhood of s' = 0. Then the family F' smoothes the point (0;s'), and after shrinking we obtain a (*) neighborhood of (0;s') where the family is sufficiently general, hence has a splitting component by step 1). Since all components of D contain 0, thus there is some component of D passing through 0, over which the degree of C-->D is greater than 1. Hence the singularity at (0;0) splits in codimension one at least in the larger family. Next we want to deduce the same fact within the original family.

Step three: By step two we may assume given an essential splitting component Z' of the enlarged family, and we intersect it with the original base space V. Since Z' contains 0, that intersection is a divisor in V also containing 0. Let Z be an irreducible component of Z'meetV, such that Z contains 0, (in fact every component does so). We claim that Z is a splitting component of N for (0;0) in the original family. If not, then over a general point s of Z there is only one singularity. But general points of Z' are dense in Z, so every point s of Z is a limit of general points s' of Z' over which there are at least two singularities. By properness, as s'-->s, the singularities over s' approach the unique singularity over s, i.e. the singularity over s splits, hence is not an odp. This violates the hypothesis that the original family over V was sufficiently general.

Thus in a sufficiently general even family, an isolated singularity at (0;0) must split if it has positive corank. **QED.**

V. Conjectures: It seems reasonable to conjecture that for g ≥ 5, the intersection N'(0) meet "Thetanull" is as simple as possible, i.e. has exactly two irreducible components, R and D. Moreover at a general (A, 'theta") in R there should be exactly one singularity, a double point of order two with local equation x1^2+...+x(g-1)^2 + x^4(g) + bx^2(g) + c = 0, hence corank one and Milnor number 3.

For a general (A, 'theta") on D there should be exactly three singularities on "theta", one odp at a point of order two, and two other distinct conjugate odps {x,-x}, again with global Milnor number = 3. It follows of course from [G-SM2] and the present work that at (A, "theta") on D-R, any singularity of 'theta" at a point of order two must be an odp.

The question of whether the total Milnor number of "theta" is in general three on both components R,D, is whether a general point of (N'(0)meetThetanull) has multiplicity three on the divisor N(0) = 2N'(0)+Thetanull, or equivalently whether both reduced varieties N'(0) and Thetanull are smooth at a general point of (N'(0)meetThetanull).

Other big questions remaining open include those relating the dimension of the singular locus with the existence of thetanulls. For instance it is known that if the dimension of sing("theta") is g-2, then thetanulls exist, since then (A,"theta") is a product ppav [Ein-Lazarsfeld]. According to Mumford [Tata vol.2], if a ppav (A,"theta") not only has thetanulls, but has a sufficiently large collection of them satisfying certain precise incidence relations relative to the Riemann pairing, then (A,"theta") is a hyperelliptic Jacobian. Hence steps toward recognizing hyperelliptic Jacobians would be to decide whether dim.sing(theta) ≥ g-3 forces the existence of a thetanull, or whether hyperelliptic Jacobians are the only component of N(g-3) contained in "Thetanull".

From the perspective of using rank as an infinitesimal version of dimension of the singular locus as originated in [A-M] and emphasized in G-SM2], one can ask whether the existence of a thetanull of rank ≤ 3 forces (A,"theta") to be a Jacobian, or whether a thetanull of rank ≤ 2 occurs only on a product. In the notation of [G-SM2] this involves the study of the components of the loci "theta^h null", where there exists a theta null of rank ≤ h.

References:

[A-M] A. Andreotti and A. Mayer, On period relations for abelian integrals on algebraic curves, Ann. Scuola Norm. Sup. Pisa 21 (1967), 189-238.

[B] A. Beauville, Prym varietioes and the Schottky problem, Invent. Math., 41 (1977), 149-196.

[C-VdG] C. Ciliberto, G. Van der Geer, Andreotti–Mayer Loci and the Schottky Problem, Documenta Mathematica 13 (2008) 453–504.

[D] O. Debarre, 1) Le lieu des varie'te's abe'liennes dont le diviseur theta est singulier a deux composantes, Ann. Sci. d l'E.N.S., 4e se'rie, tome 25, no.6 (1992), 687-708.
2) Erratum pour "Le lieu des varie'te's abe'liennes dont le diviseur th^eta est singulier a deux composantes", 15 avril 2008, http://www.dma.ens.fr/~debarre/23erratum.pdf.

[F] H. Farkas, Vanishing theta nulls and Jacobians, in The geometry of Riemann Surfaces and Abelian Varieties, III Iberoamerican Congress on Geometry in honor of professor Sevi'n Recillas Pishmish's 60th birthday, editors JK. Munoz Porras, S. Popescu, and R. Rodriguez, Contemporary Mathematics, vol. 397, Amer. Math. Soc., 2006.

[K-S] A. Kas and M. Schlessinger, On the versal deformation of a complex space with an isolated singularity, Math. Ann. 196 (1972), 23-29.

[G-SM] S. Grushevsky and R. Salvati Manni, (1) Jacobians with a vanishing theta null in genus 4, Israel J. Math. 164 (2008), 303-315; and (2) Singularities of the theta divisor at points of order two, Int. Math. Res. Not. IMRN, 2007, no.15, Art. ID, rnm 045, 15 pages.

[GR] R. Gunning and H. Rossi, Analytic Functions of Several Complex Variables, Prentice Hall, 1967.

[M] D. Mumford, Algebraic Geometry I, Complex Projective Varieties, Springer Verlag, 1976, Berlin, Heidelberg, New York.

[S-V] R. Smith and R. Varley, Deformations of isolated even double points of corank one, 2009 preprint, 6 pages.