ON PARAMETRIZING EXCEPTIONAL TANGENT CONES TO PRYM THETA DIVISORS

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Abstract. The theta divisor of a Jacobian variety is parametrized by a smooth divisor variety via the Abel map, with smooth projective linear fibers. Hence the tangent cone to a Jacobian theta divisor at any singularity is parametrized by an irreducible projective linear family of linear spaces normal to the corresponding fiber. The divisor variety X parametrizing a Prym theta divisor Ξ on the other hand, is singular over any exceptional point, hence although the fibers of the Abel Prym map are still smooth, the normal cone in X parametrizing the tangent cone of Ξ can have non linear fibers. As an illustrative example, we compute the case of a Prym variety isomorphic to the intermediate Jacobian of a cubic threefold, where the projectivized tangent cone, the threefold itself, is parametrized by the 2 parameter family of cubic surfaces cut by hyperplanes through a fixed line on the threefold. In general we show that over any very exceptional point on a Prym theta divisor, the components of the normal cone in X which dominate the tangent cone to Ξ are supported only on the sublocus of divisors which arise from the Shokurov line bundle associated to the very exceptional singularity. It is then easy to formulate the expected typical structure of the parametrization of tangent cones to Prym theta divisors according to 5 basic cases. We provide some other examples to show the diversity of structure that can actually occur, and identify some open problems. In an appendix we include the original Mumford/Kempf proof of the Riemann singularity theorem.

1. Introduction

Describing a tangent cone to a Jacobian theta divisor has three important aspects:

i) the numerical one of computing the multiplicity at a point in terms of the dimension of sections of the line bundle associated to that point,

ii) the algebraic one of giving determinantal equations for the tangent cone in terms of such sections, and

iii) the geometric one of parametrizing the tangent cone by a linear family of varieties of infinitesimal variations of divisors associated to such sections.

For Jacobian theta divisors, these descriptions were provided by Riemann [Ri], Mumford and Kempf [K1, K3], and Andreotti-Mayer [A-M] and Kempf

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[K1], respectively. In the case of Prym theta divisors, the precise multiplicity was determined relatively recently by Casalaina-Martin [CM], and consequent Pfaffian algebraic equations given by Smith and Varley [S-V6]. The geometric parametrization is still lacking in a completely precise form; in particular even for double points the problem of the rank of the quadric tangent cone is not well understood. By analogy with the work of Mark Green [G] on constructive Torelli for Jacobians, this has bearing on the constructive Torelli problem for Pryms which, by the recent counterexamples of Izadi and Lange [I-L], seems to be rather complicated and interesting. For generic Prym varieties of dimension at least 7, Debarre [D] showed that the étale double cover of curves can be reconstructed from the double points of the Prym theta divisor.

The case of Jacobians is simpler because the divisor variety provides a smooth birational resolution of the theta divisor with smooth scheme theoretic linear fibers. Consequently the tangent cones are always birationally ruled by a linear family of linear spaces of known constant dimension. E.g. a tangent cone to a Jacobian theta divisor Θ is always an irreducible rational variety. Except in the hyperelliptic case, this resolution of Θ is even a small one and the "equisingular" deformations of the Jacobian theta divisor correspond exactly to deformations of the divisor variety [S-V2, Lemma, p. 251], at least to first order.

For Pryms, the divisor variety is never birational to the theta divisor Ξ and not always smooth. The scheme theoretic fibers of the Abel Prym map are still smooth, and this implies for Pryms with smooth divisor variety, i.e. those with no exceptional singularities on Ξ, that every tangent cone is again parametrized, although not birationally, by a linear family of linear spaces of known constant dimension [S-V3]. In particular the tangent cone at a non exceptional singularity of a Prym theta divisor is always irreducible and unirational. This leaves open the problem of describing the geometric parametrization in the presence of exceptional or very exceptional singularities on Ξ.

This paper makes a beginning on the problem by providing a complete description (Theorem 1) in the (very exceptional) case of a Prym representation of the intermediate Jacobian of a cubic threefold, and giving one general result (Theorem 2) about the parametrization of the tangent cone at an arbitrary very exceptional singularity of a Prym theta divisor. Thus the typical structure of the parametrization for very exceptional singularities is displayed. Also an example is given of unexpected behavior that can occur in the exceptional but not very exceptional case.

Organization of the paper. We recall in section 2 some basic facts about the theta divisor Ξ of the Prym variety of a double cover $\widetilde{C} \to C$, and the parametrization of Ξ by the restriction of the Abel map of $\widetilde{C}$ to a special divisor variety X. Then in section 3 we compute the Abel Prym parametrization of the tangent cone to Ξ for the basic example of a very exceptional singularity on a Prym theta divisor, the unique triple point on the theta divisor of the intermediate Jacobian of a cubic threefold W. As is known [B3], the intermediate Jacobian can be represented as the Prym variety of a double cover $\widetilde{C}$ of the plane quintic discriminant curve C of a representation of the cubic threefold as a conic bundle via plane sections through a fixed general line ℓ in W.

We will show in this case, contrary to the case of a Jacobian, that if $L = \pi^*(g^2_5)$ is the line bundle on $\widetilde{C}$ pulled back from $\mathcal{O}(1) = M$ on the plane quintic C, corresponding to the triple point L on Ξ, then the normal cone to the fiber $|L|$ in X of the Abel Prym map, which maps onto the tangent cone to Ξ at L, is reducible with two components. Moreover, the only component that dominates the tangent cone to Ξ at L is supported on a proper subvariety of that fiber, in fact on the subvariety $S = \mathbb{P}(\pi^* H^0(g^2_5))$ corresponding to the pulled back sections of the very exceptional "Shokurov" line bundle $M = g^2_5$ on the plane quintic C. Contrary to the case of Jacobians, this parametrization is not by a family of linear spaces but by a linear net of cubic surfaces, sections of the threefold W by the net of hyperplanes in $\mathbb{P}^4$ passing through the distinguished line ℓ in W.

In section 4, we explain the geometry of Mumford's skew-symmetric matrix of linear forms associated to each point of a Prym theta divisor. Then in section 5 we identify a general phenomenon for very exceptional singularities. I.e. for any very exceptional singularity L on a Prym theta divisor Ξ, there is by definition a subvariety $\pi^*|M| + B$ of the fiber $|L|$ of the Abel Prym map over L, with $h^0(M) > \frac{1}{2}h^0(L)$. Assuming that $S = \pi^*|M|+B$ has maximal dimension among such M, the tangent cone to Ξ is parametrized not just by the normal cone to the fiber $|L|$ in the Prym divisor variety X, but by those components of the normal cone supported on the subvariety S. In particular, when S is not all of $|L|$, the normal cone of the divisor variety along the Abel Prym fiber over a very exceptional singularity of Ξ is reducible, contrary to the case of Jacobians.

In section 6, we formulate what the expected typical structure of the parametrization seems to be and make some remarks. For instance the situation for the cubic threefold generalizes somewhat to the case of any odd dimensional cubic hypersurface containing a linear subspace of codimension 2 in the hypersurface. We include 2 examples illustrating expected behavior and pose a few basic questions. In the final section 7 we present an example to show how some of the "expected" structure can break down; in particular a reducible normal cone can occur also at an exceptional but not very exceptional singularity.

In an appendix, we reproduce an account, following old expository notes of Kempf, of the famous "unpublished" proof of the Riemann singularity theorem, as given by Mumford and enhanced by Kempf. Throughout this paper we work over the complex numbers $\mathbb{C}$, for simplicity and uniformity.

Most of the arguments will still hold unchanged over an algebraically closed field of characteristic $\neq 2$, but we have not checked any additional details or references.

2. The parametrization of a tangent cone

2.1. **The setup.** For full details on the Abel Prym map consult [S-V3]. Let $\varphi : X \to P$ be the Abel Prym map for a connected étale double cover $\pi : \widetilde{C} \to C$ of a smooth curve C of genus g. Then the polarized Prym variety P is an abelian subvariety of the Jacobian $(\widetilde{J}, \widetilde{\Theta})$ of $\widetilde{C}$, with principal polarization Ξ defined by $2\Xi = P \cdot \widetilde{\Theta}$, and the Prym divisor variety X is the inverse image of P, hence also of Ξ, under the Abel map $\tilde{\alpha} : \widetilde{C}^{(2g-2)} \to \widetilde{\Theta}$ for $\widetilde{J}$. We denote the restriction of $\tilde{\alpha}$ to X by $\varphi : X \to \Xi$. When C is non hyperelliptic, as we assume hereafter, X is irreducible of dimension $p = dim(P) = g - 1 = \tilde{g} - g$, and φ maps X onto the theta divisor Ξ of P, with fibers isomorphic to odd dimensional projective spaces, with generic fiber dimension one. Over a point L of Ξ, the fiber of φ is isomorphic to the linear series $|L| = \mathbb{P}H^0(\widetilde{C}, L)$.

The multiplicity of Ξ at L is as small as possible, equal to $\frac{1}{2}h^0(L)$, unless L is a very exceptional singular point. Recall that L is called <u>exceptional</u> if the space $H^0(L)$ has an isotropic subspace of dimension at least 2 for the Mumford pairing

$$H^0(L) \times H^0(L) \xrightarrow{\beta} H^0(\Omega_C(\eta)),$$

and is called <u>very exceptional</u> if it has an isotropic subspace of dimension greater than $\frac{1}{2}h^0(L)$. This pairing comes from [M, p. 343]: for $s, t \in H^0(\widetilde{C}, L)$, first let $\langle s, t\rangle$ in $H^0(\Omega_{\widetilde{C}})$ denote the cup product of s and $\iota^*(t)$, by $H^0(L) \times H^0(\Omega_{\widetilde{C}} \otimes L^*) \to H^0(\Omega_{\widetilde{C}})$, where ι is the involution of $\widetilde{C}$ over C and the isomorphism $\iota^*(L) \cong \Omega_{\widetilde{C}} \otimes L^*$ is used (since the norm of L is Ω_C). Then set $\beta(s, t) = \langle s, t\rangle - \langle t, s\rangle$ in the subspace $H^0(\Omega_C(\eta))$.

The term "exceptional singularity" has been fairly standard for Mumford's case 1 in [M, p. 344]. By Casalaina-Martin's result [CM, Thm. 2, p. 164], the multiplicity of L on Ξ equals the larger of $\frac{1}{2}h^0(L)$ and the dimension of the largest isotropic subspace of $H^0(L)$.

2.2. **The parametrization induced by φ for a tangent cone to Ξ.** Let L be any point on Ξ, and $|L|$ the fiber of φ over it. Then the projective normal cone to $|L|$ in X maps onto the projective tangent cone at L to Ξ, by the map induced by blowing up the subvarieties $|L|$ and $\{L\}$ in X and Ξ respectively. Moreover, X is contained in the smooth variety $\widetilde{C}^{(2g-2)}$, and the induced map on the blowup of $|L|$ in X factors through the induced map of the blowup of $|L|$ in $\widetilde{C}^{(2g-2)}$. Since both $|L|$ and $\widetilde{C}^{(2g-2)}$ are smooth, that latter induced map is given by the differential of the Abel map on $\widetilde{C}^{(2g-2)}$. Thus every point of the normal cone to $|L|$ in X is represented by a Zariski tangent vector to $\widetilde{C}^{(2g-2)}$, and the induced map from the normal cone of $|L|$

in X to the tangent cone of Ξ at L, is the restriction of the Abel differential acting on the Zariski tangent space.

Now if L is not an exceptional singularity on Ξ, i.e. $H^0(L)$ has no isotropic subspaces of dimension 2 or more, then the tangent cone to Ξ is parametrized by the family of Zariski normal spaces to $|L|$ in X. I.e. the tangent cone in the non exceptional case is the intersection of $T_0(P)$ with the tangent cone to $\widetilde{\Theta}$ in $T_0(\widetilde{J})$, which is parametrized by the normal Zariski tangent bundle to $|L|$ in $\widetilde{C}^{(2g-2)}$. The Zariski normal bundle to $|L|$ in X is the pullback of $T_0(P)$ by the differential of the Abel map on $\widetilde{C}^{(2g-2)}$, and hence that bundle maps onto the tangent cone to Ξ by the Abel differential, i.e. the map induced on the blowup of $|L|$ in X. This is proved in our paper [S-V3, Cor. 2.8].

3. Abel Prym differential for the Prym representation of a cubic threefold W defined by a line ℓ in W

We will compute the parametrization $\mathbb{P}C_{|L|}(X) \to \mathbb{P}C_L(\Xi)$ for the beautiful case of a Prym representation of the cubic threefold, in which the unique singularity is very exceptional.

Let $\varphi : X \to P$ be the Abel Prym map for an odd double cover $\pi : \widetilde{C} \to C$ of a plane quintic curve C associated to a general line ℓ in a smooth cubic 3 fold W. Then the polarized Prym P is isomorphic to the polarized intermediate Jacobian of W. In particular X maps onto the theta divisor Ξ, a 4 fold in P with one singular point L, over which the fiber of φ is $E = |g^3_{10}| \cong |\pi^*(g^2_5))| \cong \mathbb{P}^3$, and the projective tangent cone to Ξ at L is W. A clear and complete proof of this story on the singularities of theta for the intermediate Jacobian of a cubic threefold was published by Beauville [B3]; the statement is in Mumford's paper [M, p. 348]. Some additional references are [F, p. 80], [S-V1, Ex. 6.4, p. 670], and [S-V3, §5, pp. 503-506].

Theorem 1. *If W is a smooth cubic threefold, the Abel Prym map $\varphi : X \to \Xi$ defined by a general line ℓ in W, induces a map from the reducible (2 components) projective normal cone along $|\pi^*(g^2_5)|$ in X onto the projective tangent cone at L (isomorphic to W) of Ξ. One component is a trivial $\mathbb{P}^1$ bundle over $|L| \cong \mathbb{P}^3$ which collapses onto the line ℓ, and the fibers of the other component, supported over $\pi^*|g^2_5| \cong \mathbb{P}^2$, map isomorphically to the cubic surfaces cut on W by the net of hyperplanes through ℓ in $\mathbb{P}T_0(P) \cong \mathbb{P}^4$.*

Proof. We know [S-V3, S-V5] the singular locus of X consists precisely of the subvariety $S = \pi^*|g^2_5| \cong \mathbb{P}^2$ of E. Moreover by the general theory of blowups, the projective normal cone of X along E maps surjectively onto W by the restricted differential φ_* of φ. (Indeed the differential φ_* on the projective normal space $\mathbb{P}N_E(X)$ containing this normal cone $\mathbb{P}C_E(X)$ is induced by a map on $\mathbb{P}N_E(\widetilde{C}^{(10)})$, from blowing up the ambient smooth variety $\widetilde{C}^{(10)}$ along E, and this latter map is the differential of the Abel map on $\widetilde{C}^{(10)}$.)

The Zariski tangent spaces of X have dimension 5 away from S, and dimension 7 at points of S, according to [S-V3, Cor. 2.14, p. 491]. The fibers of φ are all smooth by the Riemann-Kempf, Mattuck-Mayer theorem, and since E is the only fiber which is not isomorphic to $\mathbb{P}^1$, the kernel of the differential of φ acting on the Zariski tangent space to X has dimension one away from E and dimension 3 at points of E. Thus the image of the differential of φ has linear dimension $5 - 1 = 4$ away from E, $5 - 3 = 2$ at points of $E - S$, and $7 - 3 = 4$ at points of S.

We claim the image space of that differential is constant on $E - S$, i.e. at all points of $E - S$ the image of the differential of φ is the same 2 dimensional linear subspace of the tangent space to P at L. Equivalently the projective image of the differential of φ is the same projective line in W for every point of $E - S$. To see this, note that the differential of φ defines a morphism from $E - S$ to the Fano surface of lines in W. Since that Fano surface embeds in the intermediate Jacobian of W it does not contain any rational curves. Since $E - S$ is rationally connected, this morphism is constant. [The constancy of the map on $E - S$ in this case can also be seen directly from the skew-symmetric matrix of linear forms described in section 4.]

Thus the 4 dimensional irreducible component of the projective normal cone of X along E corresponding to the closure of the normal $\mathbb{P}^1$ bundle of $E - S$ maps by the differential of φ onto a single line in W. Since the full normal cone maps onto W there must be other components supported in S. We claim there is only one, supported on all of S.

First of all, at points of S the Zariski tangent space to X has dimension 7 and the kernel of ϕ_* has dimension 3, so the Zariski normal spaces have dimension 4. Hence each fiber of the projective normal cone at a point of S is contained in a $\mathbb{P}^3$, and ϕ_* is injective on this $\mathbb{P}^3$. Since W does not contain any copies of $\mathbb{P}^3$, the fibers of the projective normal cone at points of S have dimension at most 2. In particular, since $S \cong \mathbb{P}^2$, no component of the pure 4 dimensional projective normal cone is supported in a proper subvariety of S.

Thus if Z is the part of the projective normal cone supported in S, then every component of Z is supported on all of S, all fibers of $Z \to S$ have pure dimension 2, and each fiber injects by ϕ_* into the cubic surface cut from W by the image of the projective normal space under ϕ_*, which is isomorphic to $\mathbb{P}^3$. Since W contains no 2-planes, each of these cubic surface sections is irreducible. Hence each fiber of $Z \to S$ is an irreducible cubic surface, and by [Shaf, Ch. I, 6.3, Theorem 8, p. 77] Z itself is irreducible, i.e. there is only one irreducible component of the normal cone supported on S.

Thus the projective normal cone to E in X is pure 4 dimensional and has two irreducible components. One component is supported on $E \cong \mathbb{P}^3$ and is at least generically a $\mathbb{P}^1$ bundle that maps onto a unique line ℓ' in W. The other component is supported on $\pi^*|g^2_5| \cong \mathbb{P}^2$ and maps onto W, with each fiber of this component over a point of $\pi^*|g^2_5|$ mapping isomorphically onto a cubic surface hyperplane section of W. We will show that the component of the projective normal cone in X along $\pi^*|g^2_5|$ maps by the differential of φ to the net of cubic surface sections of W cut by the net of $\mathbb{P}^3$'s passing through the line ℓ, in particular that $\ell = \ell'$.

To see that $\ell' = \ell$, we recall the way the line ℓ is used to construct the curve C. First W in $\mathbb{P}^4$ is fibered over the $\mathbb{P}^2$ of planes through ℓ, with fibers the net of conics residual to ℓ in each plane section of W. Then C is the curve of reducible conics in this net, and $\widetilde{C}$ is the curve of components of reducible conics. The Prym canonical model of C in $\mathbb{P}^4$ is the locus of singular points of reducible conics and is mapped isomorphically to the plane quintic model by the projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^2$ with the line ℓ as center.

Under this projection, a line in $\mathbb{P}^2$ corresponds to a hyperplane in $\mathbb{P}^4$ containing ℓ. Hence each divisor of $|g^2_5| = |M|$ cut on the quintic by a line in $\mathbb{P}^2$ corresponds to the moving part of the Prym canonical divisor cut on the Prym canonical model by the corresponding hyperplane in $\mathbb{P}^4$ through ℓ. Since by Riemann Roch only one Prym canonical divisor vanishes on each divisor of the g^2_5, the net of divisors of the g^2_5 on the Prym canonical curve span the net of hyperplanes in $\mathbb{P}^4$ through ℓ. Moreover, recalling that $K + \eta = M + (M + \eta)$, the fixed part of the corresponding net of Prym canonical divisors, i.e. the unique divisor in $|M + \eta|$, spans the line ℓ.

If D is a divisor of $|M| = |g^2_5|$, and $\pi^*(D)$ is the corresponding very exceptional divisor in $|L|$, it follows from the geometric description of the Prym canonical map [S-V3, Lemma 5.2, p. 504] or [T], that the hyperplane spanned by D on the Prym canonical curve is contained in the intersection of $\mathbb{P}T_0(P)$ with the span of the divisor $\pi^*(D)$ on the canonical model of $\widetilde{C}$ in $\mathbb{P}T_0(\tilde{J})$. Moreover by the formula in [S-V3, Lemma 2.6, p. 488] this intersection equals the (projective) image of the differential of φ.

It follows that the base locus ℓ of the hyperplanes spanned by divisors of $|M|$ on the Prym canonical curve must equal the base locus ℓ' of the images of φ_* at points of $S = \pi^*|M|$ in X. One can also compute directly the image of φ_* at the unique divisor of $\pi^*|M + \eta|$ in $E - S$. Namely by the same argument as above, this image contains hence equals the span ℓ of the divisor $|M + \eta|$ on the Prym canonical curve. Since φ_* has the same image ℓ' at all points of $E - S$, we again see that $\ell = \ell'$. □

Remarks. This discussion of the parametrization $\mathbb{P}C_E(X) \to \mathbb{P}C_{\{\pi^*(g^2_5)\}}(\Xi)$ associated to a cubic threefold $W \subset \mathbb{P}^4$ and general line ℓ on it, shows that the normal cone $C_E(X)$ with its reduced structure, has multiplicity 4 at a general point of $S = \pi^*|g^2_5|$. The irreducible component of $C_E(X)$ supported over S is the family of affine cubic cones over the hyperplane sections of W through ℓ, and the component supported over all of E is smooth in this case. Indeed, with $\mathbb{P}C_E(X) \hookrightarrow E \times \mathbb{P}^4$, it was shown that the irreducible component that dominates E maps onto a line $\ell' \subset \mathbb{P}^4$ and hence is a closed subvariety of the 4-dimensional variety $E \times \ell'$. Then, since $\mathbb{P}C_E(X)$ has pure dimension $= dim(X) - 1 = 4$, this component must be (set-theoretically) equal to $E \times \ell'$.

For the line bundle $L = \pi^*(g^2_5)$ in this case, Mumford's skew-symmetric matrix of linear forms is 4×4 and can be written (see [Shok, p. 121]) with a 3×3 upper left block of 0's and 3 linearly independent linear forms in the 4th column (above the 0 in the bottom right corner). The 3 linear forms give equations for the hyperplanes in $\mathbb{P}^4$ corresponding to the pullback of sections of g^2_5 on $\mathbb{P}^2$, hence define the line $\ell \subset \mathbb{P}^4$. This base locus also equals ℓ' by Cor. 1 of the next section.

4. The geometry of Mumford's skew-symmetric matrix of linear forms

For $\varphi : X \to \Xi$ and $L \in \Xi$ we are interested in the parametrization $C_{|L|}(X) \to C_L(\Xi)$ of the tangent cone by the normal cone along the fiber. But how much of the structure is already determined at 1st order, i.e. by the Abel Prym map derivative $\varphi_* : N_{|L|}(X) \to T_L(P)$ on the normal space along the fiber? Thus, consider the diagram:

$$\begin{array}{ccccc} |L| \subset X & & & |L| \leftarrow N_{|L|}(X) \supseteq C_{|L|}(X) \\ | \quad\quad \downarrow \varphi & & \rightsquigarrow & \varphi_* \downarrow \quad\quad\quad \downarrow \\ L \in \Xi \subset P & & & T_L(P) \supset C_L(\Xi) \end{array}$$

According to Kempf and Mumford, $N_{|L|}(X) \subset |L| \times T_L(P)$ is pulled back from a universal linear algebra model, where $N_{|L|}(X)$ is embedded in $|L| \times T_L(P)$ by the product of the projection map $N_{|L|}(X) \to |L|$ and $\varphi_* : N_{|L|}(X) \to T_L(P)$. We will review a formulation of this result. Throughout we use the canonical identification of the tangent spaces $T_L(P)$ and $T_0(P)$ by translation in the Prym variety, and write whichever one seems clearest in each instance.

The skew-symmetric bilinear form $\beta : H^0(L) \times H^0(L) \to H^0(\Omega_C(\eta))$ can be represented w.r.t. a basis for $H^0(L)$ as a skew-symmetric matrix M_1 of linear forms on $T_0(P)$, since $H^0(\Omega_C(\eta)) \cong T_0^*(P)$. Thus, with $n = h^0(L)$, let $Alt(n)$ be the vector space of all $n \times n$ skew-symmetric matrices over $\mathbb{C}$. Then M_1 defines a linear map

$$T_L(P) \to Alt(n),$$

so that if we identify $|L| = \mathbb{P}^{n-1}$ (from $H^0(L) \cong \mathbb{C}^n$ = column vectors, w.r.t. to our basis) then we get a map on the product

$$|L| \times T_L(P) \xrightarrow{(ident., M_1)} \mathbb{P}^{n-1} \times Alt(n).$$

Now define

$$\mathbb{K} = \{([\sigma], A) \in \mathbb{P}^{n-1} \times Alt(n) \mid A\,\sigma = 0\},$$

where $[\sigma]$ denotes the point of $\mathbb{P}^{n-1}$ determined by a nonzero column vector $\sigma \in \mathbb{C}^n$.

Theorem (Kempf and Mumford). $N_{|L|}(X) = (ident., M_1)^{-1}(\mathbb{K})$ as subschemes of $|L| \times T_L(P)$.

Proof. We can reference Kempf [K1, Lemma, p. 183], in the language of "well-presented families of projective spaces", for the corresponding 1st order statement and proof in the Jacobian case. That is, since L is a point of the theta divisor of the Jacobian $\widetilde{J}$ of the curve $\widetilde{C}$, one has the geometry associated with the Riemann-Kempf matrix, which is an $n \times n$ matrix of linear forms on $T_0(\widetilde{J})$ and is never skew-symmetric. Then Mumford [M, pp. 342-343] showed how to restrict all the 1st order objects to the Prym case. □

Corollary 1. For $L \in \Xi$, let s be a nonzero section of the line bundle L on $\widetilde{C}$ and let $[s] \in |L| \subset X$ denote the divisor of s. In terms of a basis for $H^0(L)$, if s is expressed as a column vector and M_1 as a skew-symmetric matrix of linear forms on $T_0(P)$, then the image in $T_L(P) \cong T_0(P)$ of the fiber of $N_{|L|}(X)$ at the point $[s]$, i.e. of the Zariski normal space to $|L|$ in X at $[s]$, is the common base locus in $T_0(P)$ of the column vector of linear forms $M_1 s$. Intrinsically, the equivalent statement is that the image $\varphi_*(N_{|L|}(X)|_{[s]}) \subset T_L(P) \cong T_0(P)$ is equal to $\{v \in T_0(P) \mid \beta(\cdot, s)(v) = 0 \text{ (as an element of } H^0(L)^*)\}$.

Proof. A vector v in $T_0(P)$ is in that common base locus if and only if the column vector $(M_1 s)(v) = M_1(v)s = 0$, iff s is in the kernel of $M_1(v)$, iff $([s], M_1(v))$ is in $\mathbb{K}$, if and only if $([s], v)$ is in $N_{|L|}(X)$. □

Corollary 2. Every irreducible component of the normal space $N_{|L|}(X)$ has dimension at least $p = dim(P)$.

Proof. By the theorem, $N_{|L|}(X)$ is the inverse image of a smooth subvariety of codimension $n-1$ in a smooth variety. Hence every irreducible component of $N_{|L|}(X)$ has codimension at most $n - 1 = dim|L|$ in $|L| \times T_L(P)$, i.e. dimension $\geq p$. □

Remark. In principle the results summarized here have been completely known at least since 1974. The higher order statement in the Jacobian case is given with references in [K1, Thm., p. 183] and a proof is given in [K3, Thm. 21.4, p. 189]. The original arguments of Mumford and Kempf are summarized in the Appendix to the present paper, following [K2]. One may consult our paper [S-V6, sections 3-4] for some review and formulations. The main point of [S-V6] was to refine the known existence theory for a local inducing map from the Jacobian $\widetilde{J}$ to matrix space, to get the existence [S-V6, Thm., p. 225] of a local inducing map from the Prym P to skew-symmetric matrix space. It is the 1st order part, i.e. the derivative at L of a nonlinear map, that always has an intrinsic description as the linear map

$$T_L(P) \longrightarrow \Lambda^2(H^0(L)^*) \cong \{\text{alternating bilinear forms on } H^0(L)\}$$

dual to Mumford's linear map ([M, display just below the middle of p. 343])

$$T_0^*(P) \cong H^0(\Omega_C(\eta)) \longleftarrow \Lambda^2 H^0(L).$$

Thus, in terms of a basis for $H^0(L)$, if we identify the alternating bilinear forms on $H^0(L)$ with the skew-symmetric $n \times n$ matrices $Alt(n)$ (and also equate $T_0(Alt(n))$ with $Alt(n)$), then we say simply that Mumford's skew-symmetric matrix of linear forms on $T_0(P)$ expresses this derivative at L as $T_L(P) \to Alt(n), z \mapsto M_1(z)$, by evaluating all the entries at $z \in T_L(P) \cong T_0(P)$.

Now we look at the corresponding linear algebra diagram:

$$\begin{array}{ccccccccc} \mathbb{P}^{n-1} & \subset & \mathbb{K} & \subset & \mathbb{P}^{n-1} \times Alt(n) & & \mathbb{P}^{n-1} \leftarrow N_{\mathbb{P}^{n-1}}(\mathbb{K}) & = & C_{\mathbb{P}^{n-1}}(\mathbb{K}) \\ | & & \downarrow q & & \downarrow & \rightsquigarrow & \downarrow & & \downarrow \\ 0 & \in & \Sigma & \subset & Alt(n) & & Alt(n) & \supset & C_0(\Sigma) = \Sigma \end{array}$$

Here Σ is the locus in $Alt(n)$ consisting of all singular $n \times n$ skew-symmetric matrices. Precisely, $\Sigma \subset Alt(n)$ is the closed subscheme defined by the equation $Pf(A) = 0$, where the Pfaffian Pf is a homogeneous polynomial function of degree $n/2$ on $Alt(n)$ whose square is the determinant. Σ is a conical hypersurface (with vertex at 0) since $Pf(A)$ is homogeneous in A, and is reduced and irreducible.

The variety $\mathbb{K}$ is smooth, and its closed subscheme $\{([\sigma], 0)\} \cong \mathbb{P}^{n-1}$ is smooth, so the normal space $N_{\mathbb{P}^{n-1}}(\mathbb{K})$ is a vector bundle over $\mathbb{P}^{n-1}$ and is equal to the normal cone $C_{\mathbb{P}^{n-1}}(\mathbb{K})$ of $\mathbb{K}$ along $\mathbb{P}^{n-1}$. The tangent cone of Σ at 0 is Σ itself: $C_0(\Sigma) = \Sigma$ (under the canonical vector space identification $T_0(Alt(n)) = Alt(n)$). Similarly, the normal bundle $N_{\mathbb{P}^{n-1}}(\mathbb{K}) \subset \mathbb{P}^{n-1} \times Alt(n)$ is already exactly $\mathbb{K}$ itself.

Thus the parametrization $q : \mathbb{K} \to \Sigma, ([\sigma], A) \mapsto A$, (induced by the projection $\mathbb{P}^{n-1} \times Alt(n) \to Alt(n)$ to the 2nd factor), can also be interpreted as the parametrization $N_{\mathbb{P}^{n-1}}(\mathbb{K}) \to C_0(\Sigma)$ of the tangent cone by the normal cone (or space, or bundle) along the fiber.

In summary, the map $N_{|L|}(X) \to T_L(P)$ is equivalent to the pullback of the map q (viewed as $\mathbb{K} = N_{\mathbb{P}^{n-1}}(\mathbb{K}) \to C_0(\Sigma) = \Sigma \subset Alt(n)$) by the linear map $T_L(P) \to Alt(n)$:

$$\begin{array}{ccccc} |L| \leftarrow N_{|L|}(X) & & \mathbb{P}^{n-1} \leftarrow & \mathbb{K} \\ \varphi_* \downarrow & & & \downarrow \\ T_L(P) & \xrightarrow{M_1} & & Alt(n) \end{array}$$

In other words, the Abel Prym derivative on the normal space of X along $|L|$ is induced - by Mumford's skew-symmetric matrix of linear forms - from the completely standard (projective) "kernel fibration" $q : \mathbb{K} \to \Sigma \subset Alt(n)$. For any $A \in Alt(n)$ the fiber of q is the projectivized null space $\mathbb{P}ker(A)$, and the generic fiber over Σ is a $\mathbb{P}^1$.

Now we signal exactly what is parametrized by the normal space $N_{|L|}(X)$. The result [S-V3, Cor. 2.8, p. 489] is that $N_{|L|}(X)$ always parametrizes the restriction of the Jacobian tangent cone $C_L(\widetilde{\Theta})$, to the Prym tangent space $T_L(P)$; that is,

$$\varphi_*(N_{|L|}(X)) = C_L(\widetilde{\Theta}) \cap T_L(P).$$

It follows that either $C_L(\widetilde{\Theta}) \cap T_L(P) = C_L(\Xi)$ as sets or else $C_L(\widetilde{\Theta}) \supset T_L(P)$. In the first case (that is, when $C_L(\widetilde{\Theta}) \cap T_L(P) = C_L(\Xi)$ as sets), then the scheme $C_L(\Xi)$ is defined as expected by an equation whose square is an equation for the scheme $C_L(\widetilde{\Theta}) \cap T_L(P)$, and we say that "RST holds" at L. The main theorem of [S-V5] is that the second case occurs (that is, the containment $C_L(\widetilde{\Theta}) \supset T_L(P)$ holds, so that $\varphi_*(N_{|L|}(X)) = T_L(P)$) if and only if L is a very exceptional point of Ξ (as defined on p. 4 above).

Finally, note that there is a difference between analyzing:

(a) all $n \times n$ skew-symmetric matrices of linear forms, perhaps subjected to some additional properties (such as that of [S-V5, Lemma 2.3(iii), p. 451]), and

(b) those that actually occur as "M_1" for an étale double cover of curves and L on Ξ.

In other words, certain phenomena that could happen in case (a) might be prevented in case (b) by some property coming from the curve theory.

Question. Which skew-symmetric $n \times n$ matrices of linear forms can occur?

Question. For $L \in \Xi$, what are the possible subvarieties of $|L| \subset X$ over which the irreducible components of the normal space $N_{|L|}(X)$ of X along $|L|$ are supported?

Remark. The dimension of the normal vector spaces (the fibers of $N_{|L|}(X) \to |L|$) can vary over $|L|$ since it goes up at any exceptional divisor D in $|L|$. Let us record the most elementary dimension estimate to ensure irreducibility of the normal cone $C_{|L|}(X) \subset N_{|L|}(X)$. Consider the (locally closed) locus $I_k(L)$ consisting of all divisors $D \in |L|$ for which the maximal isotropic subspace of β through D has dimension exactly $k-1$ (as projective linear subspace of $|L|$). Then for $D \in I_k(L)$, we know [S-V3] that $dim(T_D(X)) = p + (k-1)$. Therefore the fiber of the normal space $N_{|L|}(X)$ over such a point D has dimension $p + (k-1) - (n-1) = p - (n-k)$. Since the normal cone $C_{|L|}(X)$ has pure dimension p and the fibers of the normal cone are contained in the fibers of the normal space, it follows that as long as $dim(I_k(L)) < n-k$ there cannot be an irreducible component of the normal cone that dominates an irreducible component of $I_k(L)$ (i.e. that lies over an irreducible component Z of the closure of $I_k(L)$ and maps onto Z). In fact since all components of $N_{|L|}(X)$ have dimension $\geq p$, there is also no component of the normal space dominating $I_k(L)$ when $dim(I_k(L)) < n-k$. For instance, taking $k = 2$, unless there are at least ∞^{n-3} exceptional pencils in $|L|$ there is not a component of $C_{|L|}(X)$ dominating a component of $I_2(L) \subset |L|$. Thus if for all k with $2 \le k \le n$, we have $dim(I_k(L)) < n-k$, then the normal cone $C_{|L|}(X)$ is irreducible and equals the normal space $N_{|L|}(X)$.

5. General structure of the very exceptional case

As before, let (P, Ξ) be the Prym variety of a connected étale double cover $\pi : \widetilde{C} \to C$ of a connected, nonyperelliptic smooth curve of genus $g \ge 3$. Thus $p = dim(P) = g - 1$ and for any point $L \in \Xi$, the tangent cone $C_L(\Xi) \subset T_L(P)$ is a purely $p-1$ dimensional conical affine subvariety of the p dimensional vector space $T_L(P) \cong T_0(P)$. Now let L on Ξ be a very exceptional or "Shokurov" singularity on Ξ; i.e. assume there exists a subspace of $H^0(L)$ which has dimension more than $\frac{1}{2}h^0(L)$ and which is isotropic for the Mumford pairing $\beta : H^0(L) \times H^0(L) \to H^0(\Omega_C(\eta))$. If it is maximal isotropic, then it has form $\pi^*(H^0(M)) \cdot u$, where M is the unique Shokurov line bundle on C for L, and the divisor (u) has no invariant part [S-V5].

Theorem 2. *Let $L \in \Xi$ be a very exceptional singularity and let $\pi^*(H^0(M)) \cdot u \subset H^0(L)$ be the unique maximal isotropic subspace of dimension $h^0(M) > \frac{1}{2}h^0(L)$. Then the tangent cone $C_L(\Xi)$ is the image of those irreducible components of the normal cone $C_{|L|}(X)$ that lie over $\pi^*|M| + B \subset |L|$. More precisely, the union, over all divisors in $|L| - (\pi^*|M| + B)$, of the images in $T_L(P)$ of the differential of the Abel Prym map $\varphi : X \to \Xi$, cannot have dimension more than $p-3$; in particular this union cannot contain a $p-1$ dimensional component of the tangent cone $C_L(\Xi)$ of Ξ at L.*

Proof. We may as well assume that the subspace $\pi^*(H^0(M)) \cdot u$ has dimension $m = h^0(M)$ strictly between $\frac{1}{2}h^0(L)$ and $h^0(L)$, since otherwise $|L| - (\pi^*|M| + B)$ is empty and the result holds vacuously. If s is any section in $H^0(L) - \pi^* H^0(M) \cdot u$, then by the RRT of Mattuck and Mayer [M-M], the image of the differential of the Abel map on $\widetilde{C}^{(\tilde{g}-1)}$ at the divisor $s = 0$, equals the "(affine) span of that divisor" in $T_0(\widetilde{J})$, i.e. equals the base locus of those canonical differentials on $\widetilde{C}$ vanishing on that divisor. In particular it lies in the base locus of any canonical differential with s as a (cup product) factor. Now intersect that base locus with $T_0(P)$, and we get the base locus of the restriction of those differentials. Since the symmetric part of those differentials vanishes upon that restriction, the resulting subspace equals the base locus on $T_0(P)$ of the skew symmetrization of those differentials. In particular if $v_1, ..., v_m$ is a basis of $H^0(M)$, then the restriction of the span of the divisor $s = 0$ lies in the base locus of $\beta(s, u \cdot v_1), ..., \beta(s, u \cdot v_m)$, where (from p. 4 above) $\beta(s, u \cdot v_j) = \langle s, u \cdot v_j \rangle - \langle u \cdot v_j, s \rangle = s \cdot \iota^*(u \cdot v_1) - s \cdot \iota^*(u \cdot v_1)$ for each $j = 1, ...m$.

Claim. These restricted Prym differentials are linearly independent.

Proof of Claim. If not, some non trivial linear combination of the $\{u \cdot v_j\}$ pairs to zero with s under β. Then the isotropic subspace spanned by s and this non zero section of $H^0(L)$ has non trivial intersection with the maximal isotropic subspace $\pi^* H^0(M) \cdot u$, a contradiction. I.e. their union would span an even larger isotropic subspace, by results in [S-V5, Lemma 2.3(iii)]. □

Since the intersection with $T_0(P)$ of the image of the Abel differential is in the base locus of these m independent Prym differentials, that intersection has codimension at least m in $T_0(P)$.

Next we note that since the variety X is the inverse image of P under the Abel map, its tangent spaces at every point are inverse images of $T_0(P)$ under the Abel differential. Hence the intersection with $T_0(P)$ of the image of the Abel differential equals the image of the differential of the restricted Abel map, i.e. of the Abel Prym differential.

Now if we consider the divisor $s + t = 0$, where again s is any section in $H^0(L) - \pi^* H^0(M) \cdot u$, and t is any section of $\pi^* H^0(M) \cdot u$, then t pairs to zero with every $u \cdot v_j$ under β, so the base locus of $\beta(s, u \cdot v_1), ..., \beta(s, u \cdot v_m)$ equals the base locus of $\beta(s + t, u \cdot v_1), ..., \beta s + t, u \cdot v_m)$. Hence we get the same constraint on the restricted image of the differential, i.e. the image of the differentials of Abel Prym at the points $s = 0$ and $s + t = 0$ lie in exactly the same linear subspace of codimension m in $T_0(P)$. Hence the union of the linear images over all divisors of sections in $H^0(L) - \pi^* H^0(M) \cdot u$, has dimension at most equal to $(p - m) + dim\, \mathbb{P}(H^0(L)/\pi^* H^0(M) \cdot u) = (p - m) + dim(H^0(L)/\pi^* H^0(M) \cdot u) - 1$. Since $H^0(M)$ has dimension more than $\frac{1}{2}$ of $h^0(L)$, this equals at most $\frac{1}{2} h^0(L) - 2 + p - m$. Since $\frac{1}{2} h^0(L) < m$, this is at most $p - 3$, (linear dimension). Thus the image of the projectivized linear normal cones at points of $|L| - (\pi^*|M| + B)$, also has codimension at least 3 in $T_0(P)$, and the same holds for the images in $\mathbb{P}T_0(P)$ of the projective normal cones at points of $|L| - (\pi^*|M| + B)$.

Consequently the normal cones supported on the subvariety of $|L|$ in X of very exceptional divisors $\pi^*|M| + B$, must surject onto the tangent cone to Ξ. □

6. The expected typical structure and examples

6.1. **The simplest possible structure for the parametrization.** The different types of points L on a Prym theta divisor Ξ can be listed as follows:

(1) smooth point
(2) nonexceptional singular point
(3) exceptional but not very exceptional singular point
(4) very exceptional but not totally exceptional singular point
(5) totally exceptional singular point

Here, "totally exceptional" means that $H^0(L) = \pi^* H^0(M) \cdot u$, when all the sections of L on $\widetilde{C}$ are pulled back from C, up to one common multiplication upstairs (by a section of $L \otimes \pi^*(M^*)$).

In particular, we are making a subdivision of the class of singularities traditionally known as stable. Recall that a singular point L on the theta divisor of the Prym variety of an étale double cover $\widetilde{C} \to C$ is called stable if $h^0(\widetilde{C}, L) \geq 4$. Hence cases (2), (3), (4) above are all stable, and L in case (5) is non stable if and only if $h^0(L) = 2$.

For each of the 5 cases we review briefly the properties that are known to hold. Then for the exceptional cases (3,4,5) we will formulate the simplest structure that the parametrization $C_{|L|}(X) \to C_L(\Xi)$ could possibly have.

Consider $L \in \Xi$, and set $n = h^0(L)$ so that $|L| \cong \mathbb{P}^{n-1}$. In every case there is a surjective map $C_{|L|}(X) \to C_L(\Xi)$ induced by the Abel Prym differential φ_*.

(1) If L is a smooth point of Ξ, then $n = 2$, $X \to \Xi$ is a $\mathbb{P}^1$-bundle over a neighborhood of L, and the divisor variety X is smooth along $|L|$. $C_{|L|}(X) = N_{|L|}(X)$ is a vector bundle of rank $p - 1$ over $|L|$ (and hence is smooth, irreducible), and the normal bundle $N_{|L|}(X)$ maps onto the tangent cone $C_L(\Xi)$, which is a hyperplane $H \subset T_L(P)$. It follows that the normal bundle $N_{|L|}(X) \subset |L| \times T_L(P)$ is equal to $|L| \times H$, and hence is trivial over $|L|$.

(2) If L is a nonexceptional singular point of Ξ, then $n \geq 4$, the divisor variety X is smooth along $|L|$, $C_{|L|}(X) = N_{|L|}(X)$ is a vector bundle of rank $p-(n-1)$ over $|L|$ (and hence is smooth, irreducible), and the normal bundle $N_{|L|}(X)$ maps onto the tangent cone $C_L(\Xi)$. Thus $C_L(\Xi)$ is swept out by a family of linear spaces (all of the same dimension) parametrized by $|L|$.

(3) If L is an exceptional but not very exceptional singular point of Ξ, then $n \geq 4$, and the divisor variety X has some singularities in $|L|$ while the general point of $|L|$ is smooth on X. Thus $N_{|L|}(X)$ fails to be a vector bundle over $|L|$, but the fiber over a generic point does have dimension $p - (n - 1)$. The normal space $N_{|L|}(X)$ maps onto the tangent cone $C_L(\Xi)$. Therefore, as in the previous cases (1,2), Mumford's Pfaffian equation of degree $n/2$ defines $C_L(\Xi) \subset T_L(P)$ and "RST" holds at L. Surprisingly however, the unique component of $N_{|L|}(X)$, and of $C_{|L|}(X)$, dominating $|L|$, may fail to surject onto $C_L(\Xi)$; see section 7 below for an example.

(4) If L is a very exceptional but not totally exceptional singular point of Ξ, then $n \geq 4$, the divisor variety X is singular (at least) along a distinguished projective linear subspace $S \subset |L|$ of dimension $m - 1$ with $n/2 < m < n$, and X is smooth at a general point of $|L|$. The normal cone $C_{|L|}(X)$ is reducible. The unique irreducible component dominating $|L|$ does not map onto an irreducible component of $C_L(\Xi)$. Indeed every irreducible component of $C_L(\Xi)$ is dominated by some component of $C_{|L|}(X)$ which is supported over S. In particular, there is at least one irreducible component supported over S (dominating either all of S or a subvariety)

that maps onto an irreducible component of $C_L(\Xi)$. Furthermore, any irreducible component of $C_{|L|}(X)$ supported over S is properly contained in an irreducible component of $N_{|L|}(X)$.

(5) If L is a totally exceptional singular point of Ξ, then X is singular everywhere along $|L|$, the fibers of $N_{|L|}(X)$ over $|L|$ all have dimension p, and it follows that $N_{|L|}(X)$ is the trivial vector bundle $|L| \times T_L(P)$. (Indeed, $N_{|L|}(X) \hookrightarrow |L| \times T_L(P)$ is always a closed subscheme and so equality must hold in this case.)

Now what is the expected typical structure in cases (3,4,5), unless some degeneracy occurs?

In (3) the simplest possibility is for $N_{|L|}(X)$ to be irreducible, and then it would follow that $C_{|L|}(X) = N_{|L|}(X)$ and sweeps out $C_L(\Xi)$ by a family of linear spaces (not all of the same dimension) parametrized by $|L|$.

In (4) the simplest possibility is for $C_L(X)$ to have 2 irreducible components, one dominating $|L|$ and the other dominating S, with the latter component sweeping out $C_L(\Xi)$ by a family of $p-(m-1)$ dimensional subvarieties parametrized by S.

In (5) the simplest possibility is for $C_L(X)$ to be irreducible and sweep out $C_L(\Xi)$ by a family of $p-(n-1)$ dimensional varieties parametrized by $|L|$.

Question. For an exceptional (or very exceptional) singular point L on a Prym theta divisor, which subvarieties of $|L|$ are dominated by the irreducible components of the normal cone $C_{|L|}(X)$ of the divisor variety X? That is, identify the "distinguished subvarieties" (in the sense of [F, Def. 6.1.2, pp. 94-95]) of the linear series $|L| \subset X$.

Notation. For any very exceptional singular point $L \in \Xi$, with the standard unique expression $L = \pi^*(M)(B)$, we use the pair (m,n) to indicate that $m = h^0(M)$ and $n = h^0(L)$. Thus $n \geq 2$ is even, $\frac{n}{2} < m \leq n$, and m is the largest dimension of an isotropic subspace of $H^0(L)$ for the pairing $\beta : H^0(L) \times H^0(L) \to T_0^*(P)$.

Question. What is the basic geometry associated with each numerical type (m,n) of a very exceptional singularity, and what degeneracies can occur?

Remark on $(n-1,n)$. In this case X must be smooth at every point of $|L| - (\pi^*|M| + B)$. (Indeed, following [S-V6], if $D \in |L|$ is a singular point of X then there is at least an isotropic $\mathbb{P}^1$ through D. But a line in $|L| \cong \mathbb{P}^{n-1}$ must meet $\pi^*|M| + B \cong \mathbb{P}^{n-2}$, and hence in fact be contained $\pi^*|M| + B$.) Consider the irreducible component $\mathcal{C}_1$ of the normal cone $C_{|L|}(X)$ that dominates $|L|$, and the map from $\mathcal{C}_1$ to $T_L(P)$. By the arguments in the last part of the proof of Theorem 2 in the previous section, for each smooth point D of X in $|L|$, the linear map from the fiber over D in $\mathcal{C}_1$ to $T_L(P)$ is an isomorphism onto a codimension $n-1$ subspace $V \subset T_L(P)$ and in this case the image subspace V cannot vary with D. It follows that

$\mathcal{C}_1 = |L| \times V \subset |L| \times T_L(P)$, generalizing the observed behavior for the cubic 3-fold case (for which $\mathbb{P}(V)$ is the line ℓ on the cubic 3-fold in $\mathbb{P}^4$).

Remarks on $(3, 4)$. It is natural to ask for some other cases in which $(3, 4)$ occurs, besides that of intermediate Jacobians of cubic threefolds. Following [B2, Rem. 6.27, pp. 377-378], let $W \subset \mathbb{P}^{d-1}$ be an odd dimensional cubic hypersurface containing a linear subspace $\mathbb{P}^{d-4}$, with $d \geq 5$. For $d = 5$ this is the cubic threefold case, with a marked line. But for $d \geq 7$ such a cubic hypersurface is singular. So assume that the blow up $\widetilde{W}$ of W along the $\mathbb{P}^{d-4}$ subvariety is smooth, and moreover that the degree d discriminant curve $C \subset \mathbb{P}^2$ is smooth. Then [B2] there is an étale double cover $\pi : \widetilde{C} \to C$ for which the Prym variety is isomorphic to the intermediate Jacobian of $\widetilde{W}$.

We have undertaken a preliminary investigation for cubic fivefolds $\mathbb{P}^3 \subset W \subset \mathbb{P}^6$, i.e. $d = 7$. The theta divisor then has an irreducible 5 dimensional family of very exceptional triple points, expressible as $\pi^*(g^2_7) + B$ where B lies over a divisor in the linear series $|\mathcal{O}_C(2)|$ cut by conics on the plane septic curve C. We hope to obtain a Torelli result by using the tangent cone at a general such triple point.

Question. For the Prym variety (P, Ξ) of a connected étale double cover $\pi : \widetilde{C} \to C$ of a nonhyperelliptic smooth curve C, are all the tangent cones of Ξ irreducible?

Remark. Casalaina-Martin [CM, Cor. 6.2.4, p. 200] has shown that the Prym canonical model of C (assumed nonhyperelliptic) is always contained in the projectivized tangent cone $\mathbb{P}C_L(\Xi)$ (translated to Prym canonical space $PT_0(P) \cong \mathbb{P}^{p-1}$) when $h^0(L) \geq 4$, and never in the $(2, 2)$ case.

6.2. **Examples of nondegeneracy.**

6.2.1. *Example (irreducible normal cone over an exceptional singularity).* Consider a nonhyperelliptic smooth curve C with distinct theta characteristics M_1 and M_2 such that $|M_1|$ and $|M_2|$ are base point free pencils. Let $\eta = M_1 \otimes M_2^*$ and let $\pi : \widetilde{C} \to C$ be the corresponding étale double cover. Then the line bundle $L = \pi^*(M_1) = \pi^*(M_2)$ on $\widetilde{C}$ defines an exceptional singularity of the Prym theta divisor $\Xi \subset P$. $H^0(L) \cong H^0(M_1) \oplus H^0(M_2)$ is 4-dimensional, and forming a basis for $H^0(L)$ by using a basis for each $H^0(M_i)$, Mumford's 4×4 skew-symmetric matrix of linear forms has a 2×2 block of 0's in the upper left and lower right. Since the upper right 2×2 block is formed from the cup product $H^0(M_1) \times H^0(M_2) \to H^0(\Omega_C(\eta))$, the linear forms appearing as the 4 entries there are linearly independent (since $|M_1|$ and $|M_2|$ are distinct base point free pencils). It follows that the only singular points in $|L| \cong \mathbb{P}^3$ of the divisor variety X are the points of the 2 lines $\pi^*|M_i|$. By the elementary dimension estimate in the Remark at the end of section 4, the normal cone $C_{|L|}(X)$ is irreducible of dimension $p = dim(P)$ and is set-theoretically equal to the normal space $N_{|L|}(X)$.

6.2.2. *Example (variable image from the smooth points over a very exceptional singularity).* We will construct a very exceptional singularity of numerical type $(4, 6)$ to show that (in contrast to the $(n-1, n)$ case) it is possible for the image subspace of the Abel Prym differential on the fibers of $N_{|L|}(X)$ over the smooth points of X in $|L|$, to vary in $T_L(P)$. For the example we will produce a nonhyperelliptic smooth curve C having vanishing theta nulls M_1, M_2 such that $h^0(M_1) = 4$ and $|M_2|$ is a base point free pencil. Then with $\eta = M_1 \otimes M_2^*$ and $L = \pi^*(M_1) = \pi^*(M_2)$ on $\widetilde{C}$, we will get similarly to the previous example, a 6×6 skew-symmetric matrix of linear forms with a 4×4 block of 0's in the upper left and a 2×2 block of 0's in the lower right. We claim that the 8 linear forms appearing in the 4×2 upper right block are linearly independent.

Thus we need to check that we get maximal rank $(4 \cdot 2 = 8)$ for the cup product $H^0(M_1) \otimes H^0(M_2) \to H^0(M_1 \otimes M_2)$. Starting with the base point free pencil $|M_2|$ on C, consider the resulting surjection $\mathcal{O}_C \otimes H^0(M_2) \to M_2$ from a trivial rank 2 vector bundle to the line bundle M_2; cf. [ACGH, p. 126]. The kernel line bundle is isomorphic to M_2^*, so tensoring with the line bundle M_1 and taking global sections we get:

$$0 \to H^0(M_1 \otimes M_2^*) \to H^0(M_1) \otimes H^0(M_2) \to H^0(M_1 \otimes M_2) \to \ldots$$

Hence if $M_1 \not\cong M_2$ has the same degree as M_2, then $H^0(M_1 \otimes M_2^*) = 0$ and the cup product linear map is injective.

One way to carry out the construction is as follows. Take any nonhyperelliptic smooth curve C_0 of genus 4 with a vanishing theta null M. Then we will choose an appropriate order 8 subgroup $G = \langle \eta_0, \alpha, \beta \rangle$ of 1/2-periods on C in order to carry out the construction. Namely, reducing an integral symplectic basis for $H_1(C, \mathbb{Z})$ modulo 2 to equate $H_1(C, \mathbb{Z}/(2)) \cong (\mathbb{Z}/(2))^g \times (\mathbb{Z}/(2))^g$ and express each 1/2-period as two rows of 0's and 1's, let

$$\eta_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \alpha = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Of course there do exist nonhyperelliptic smooth curves of genus 4 with a vanishing theta null. What is convenient is that for such a curve C_0 there is exactly 1 vanishing theta null M (even theta characteristic with $h^0 > 0$), $|M|$ is a base point free pencil, and all the odd theta characteristics are nonsingular (i.e. $h^0 = 1$). Now consider the 8 distinct theta characteristics $\{M(\sigma) \mid \sigma \in G\}$. Four of them are even:

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

and four of them are odd:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Now pass to the étale double cover $\pi_0 : C_1 \to C_0$ defined by η_0, and pull back G and the theta characteristics $\{M(\sigma)\}$. Thus consider on C_1 the order 4 pulled-back group $\widetilde{G}$ of 1/2-periods and the four pulled-back theta characteristics. Thus $\widetilde{G}$ is generated by $\eta_1 = \pi_1^*(\alpha)$ and $\tilde{\beta} = \pi_1^*(\beta)$, and we label the theta characteristics $\widetilde{A} = \pi_1^*(A)$ and $\widetilde{M} = \pi_1^*(M)$.

Now we have:

$$h^0(\widetilde{A}) = 2,\ h^0(\widetilde{A}(\eta_1)) = 2,\ h^0(\widetilde{M}) = 2,\ h^0(\widetilde{M}(\eta_1)) = 0.$$

Next pass to the étale double cover $\pi_1 : C = C_2 \to C_1$ defined by η_1. Let $M_1 = \pi_2^*(\widetilde{A})$, $M_2 = \pi_2^*(\widetilde{M})$, and let $\eta = \pi_2^*(\tilde{\beta})$. We now have $h^0(M_1) = 4, h^0(M_2) = 2$, and $M_1(\eta) = M_2$.

So finally for $\pi : \widetilde{C} \to C$ defined by η, the line bundle $L = \pi^*(M_1)$ on $\widetilde{C}$ has a 6×6 skew-symmetric matrix of linear forms with a 4×4 block of 0's in the upper left, a 2×2 block of 0's in the lower right, and 8 linearly independent linear forms in the 4×2 upper right block.

It remains to explain why it follows that the images vary over the smooth points. Choose a basis $\{s_0, ..., s_3\}$ for $H^0(C, M_1)$, and a basis $\{s_4, s_5\}$ of $H^0(C, M_2)$, and consider $\{s_0, ..., s_5\}$ as a basis for $H^0(\widetilde{C}, L)$, (identifying the sections with their pullbacks). Denote the corresponding coordinates on $|L| \cong \mathbb{P}^5$ by $z_0, ..., z_5$. Then $\pi^*(|M_1|) = \{z_4 = z_5 = 0\} \cong \mathbb{P}^3$ is the largest linear subspace of $|L| \cap Sing(X)$. Hence that intersection cannot be all of $\mathbb{P}^5$, and thus has dimension at most 4. Then the smooth points of X are dense in $|L| - \pi^*(|M_1|) \cong \mathbb{P}^5 - \mathbb{P}^3 = \{(z_4, z_5) \neq (0,0)\}$. In particular the projection $\mathbb{P}^5 - \mathbb{P}^3 \to \mathbb{P}^1$, taking $(z_0, ..., z_5)$ to (z_4, z_5) is not constant on points of $|L| - Sing(X)$.

If we denote the last two columns of the Mumford-Kempf matrix of linear forms by v and w, then at every point of $|L| - \pi^*(|M_1|)$, the image of φ_* has codimension at most 5 in $T_L(P)$, and is contained in the codimension 4 base locus of the entries of the column of linear forms: $z_4 v + z_5 w$. If (z_4, z_5) and $(\tilde{z}_4, \tilde{z}_5)$ are not proportional, the intersection of the base loci of $z_4 v + z_5 w$ and of $\tilde{z}_4 v + \tilde{z}_5 w$ has codimension 8, hence cannot contain the image of φ_* at any point of $|L| - \pi^*(|M_1|)$.

Hence if D, E are points of $|L|$ that are smooth on X and have last two coordinates (z_4, z_5) and $(\tilde{z}_4, \tilde{z}_5)$ that are not proportional, then the images of φ_* at D and E are distinct. In fact a point of $|L|$ is smooth on X if and only if the image of φ_* has codimension 5, so we can choose D and E as the points with coordinates $(1, 0, 0, 0, 0, 1)$ and $(1, 0, 0, 0, 1, 0)$ in $|L| \cong \mathbb{P}^5$. By similar reasoning one can see that the intersection $|L| \cap Sing(X)$ consists of just two components, the union of $\pi^*(|M_1|) : \{z_4 = z_5 = 0\} \cong \mathbb{P}^3$, and $\pi^*(|M_2|) : \{z_0 = ... = z_3 = 0\} \cong \mathbb{P}^1$.

7. A degenerate example

In this section we show by example that for an exceptional, but not very exceptional, singular point L of a Prym theta divisor Ξ the unique irreducible component of the normal cone $C_{|L|}(X)$ of the divisor variety X that dominates $|L|$ need not always map onto an irreducible component of the tangent cone $C_L(\Xi)$. In particular, the normal cone $C_{|L|}(X)$ can be reducible in case (3).

First we describe the simplest possible skew symmetric matrix of linear forms that would lead to this behavior. Then we show that this matrix can occur; the example is constructed from a rather special kind of trigonal genus 6 smooth curve C and étale double cover.

Proposition 1. *Let $\pi : \widetilde{C} \to C$ be a connected étale double cover of a nonhyperelliptic smooth projective curve C of genus g. Consider the divisor variety $X \subset \widetilde{C}^{(2g-2)}$ and the parametrization $\varphi : X \to \Xi$ of the Prym theta divisor $\Xi \subset P$. Assume that $L \in \Xi$ is a point with $h^0(\widetilde{C}, L) = 4$ for which Mumford's skew symmetric matrix of linear forms can be expressed as:*

$$\begin{pmatrix} 0 & 0 & x & y \\ 0 & 0 & y & z \\ -x & -y & 0 & 0 \\ -y & -z & 0 & 0 \end{pmatrix}, \text{ with } x, y, z \text{ linearly independent on } T_L(P).$$

Then L is an exceptional singular point of Ξ and the following properties hold.

(a) The tangent cone $C_L(\Xi)$ has Pfaffian equation $xz - y^2 = 0$ and L is a rank 3 double point on Ξ at which RST holds. In particular, L is not a very exceptional singular point.

(b) The normal cone $C_{|L|}(X)$ has exactly 2 irreducible components: $\mathcal{C}_1$ supported over all of $|L| \cong \mathbb{P}^3$ and $\mathcal{C}_2$ supported over a smooth quadric surface $Q \subset \mathbb{P}^3$. Now consider the mapping from each of these two p-dimensional varieties $\mathcal{C}_1, \mathcal{C}_2$ to $C_L(\Xi) \subset T_L(P) \cong \mathbb{C}^p$ induced by restriction of the surjective morphism $C_{|L|}(X) \to C_L(\Xi)$. (We have set $p = g - 1$ as usual.)

(1) $\mathcal{C}_1 \subset |L| \times \mathbb{C}^p$ is equal to the product of $|L|$ and a codimension 3 vector subspace $E \subset \mathbb{C}^p$, so that the map from $\mathcal{C}_1$ to $\mathbb{C}^p$ is projection to the 2nd factor of the product. In particular, the image in $T_L(P)$ of the injections from the fibers in $\mathcal{C}_1$ over the points $D \in |L|$ does not vary with D.

(2) The map from $\mathcal{C}_2$ is onto $C_L(\Xi)$.

Proof. First, L is an exceptional singular point by [S-V5, Lemma 2.2 (ii), p. 450]. More precisely, if $\{s_1, ..., s_4\}$ is any basis for $H^0(\widetilde{C}, L)$ with respect to which the skew-symmetric 4×4 matrix of linear forms has the form displayed above in the assumption, then the span of the 1st two basis vectors $\{s_1, s_2\}$ is visibly an isotropic subspace W for β and so there is a line bundle M on C and a vector subspace $\Lambda \subset H^0(C, M)$ such that $L \cong \pi^*(M)(\mathcal{B})$, for some divisor $\mathcal{B} \geq 0$ on $\widetilde{C}$, and $W = \pi^*(\Lambda) \cdot u$ where $div(u) = \mathcal{B}$. Thus $h^0(M) \geq 2$ and L is an exceptional singularity of Ξ. (In fact, $W \subset H^0(L)$ is a maximal isotropic subspace for β since the linear forms x, y in the first row are linearly independent and y, z in the second row are linearly independent. Thus $h^0(M) = 2$; cf. [S-V5, Lemma 2.3 (ii), p. 451]. Similarly, for the span W_2 of the last two basis vectors $\{s_3, s_4\}$, there exists another line bundle M_2 on C with $h^0(M_2) = 2$ and $L \cong \pi^*(M_2)(\mathcal{B}_2)$, $W_2 = \pi^*(H^0(C, M_2)) \cdot u_2$.)

Next, the conclusions stated in part (a) are immediate. Namely, the Pfaffian of the skew-symmetric 4×4 matrix is $xz - y^2$, which is a rank 3 quadratic form on $T_L(P) \cong \mathbb{C}^p$. In particular, since the Pfaffian polynomial is not identically zero, it is an equation for the tangent cone $C_L(\Xi)$, RST holds, and L is not a very exceptional singularity of Ξ.

Finally, everything in part (b) will follow by direct computation with the matrix of linear forms. With respect to the basis for $H^0(L)$, consider the point $D = (a, b, c, d) \in \mathbb{P}^3 \cong |L|$. Then the 4 linear forms appearing in the column vector

$$\begin{pmatrix} 0 & 0 & x & y \\ 0 & 0 & y & z \\ -x & -y & 0 & 0 \\ -y & -z & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} cx + dy \\ cy + dz \\ -ax - by \\ -ay - bz \end{pmatrix}$$

define in $T_L(P) \cong T_0(P) \cong \mathbb{C}^p$ the image subspace $\varphi_*(T_D(X))$ of the Abel Prym differential at D.

The span of the linear forms in this column vector is not all of $\mathbb{C}x + \mathbb{C}y + \mathbb{C}z$ if and only if $ad - bc = 0$. Indeed, expressing the 4 linear forms in terms of the linearly independent ones x, y, z gives the matrix $\begin{pmatrix} c & d & 0 \\ 0 & c & d \\ -a & -b & 0 \\ 0 & -a & -b \end{pmatrix}$, and the rank of this 4×3 matrix is < 3 iff $ad - bc = 0$.

Let $\mathcal{C}_1$ be the irreducible component of the normal cone $C_{|L|}(\Xi)$ that dominates $|L|$. Since for $ad - bc \neq 0$, the image $\varphi_*(T_D(X)) \subset T_L(P)$ is constantly the codimension 3 subspace E defined by x, y, z, this component $\mathcal{C}_1$ is $|L| \times E$ and does not map onto an irreducible component of the tangent cone $C_L(\Xi)$. Therefore, there must be another irreducible component of the normal cone that maps onto an irreducible component of $C_L(\Xi)$.

Now when the point D of $|L| \cong \mathbb{P}^3$ lies on the smooth quadric surface $Q : ad - bc = 0$, the 4 linear forms span only a 2-dimensional vector space of linear forms, so $\varphi_*(T_D(X)) \subset T_L(P)$ has codimension 2 in the p-dimensional vector space $T_L(P)$. Thus over Q the fibers of the normal space $N_{|L|}(X)$ are all (p-2)-dimensional, so there is an irreducible p-dimensional component $\mathcal{C}_2$ of $N_{|L|}(X)$ lying over and dominating $Q \subset |L|$ (forming a rank p-2 vector bundle over Q). But the divisor variety X is irreducible p-dimensional, and therefore the normal cone $C_{|L|}(X)$ of X along $|L|$ has pure dimension p. Since we know that there is at least one irreducible component of $C_{|L|}(X)$ besides $\mathcal{C}_1$, and $\mathcal{C}_2$ accounts for all of $N_{|L|}(X)$ residually to $\mathcal{C}_1$, then $\mathcal{C}_2$ must be an irreducible component of the normal cone and the only one besides $\mathcal{C}_1$.

Having proved the statements we made about the general structure of $\mathcal{C}_2$ with its surjective map to $C_L(\Xi)$, we note incidentally the following explicit structure of the map. If we express a point $(a, b, c, d) \in Q$ (the rank 4 quadric surface $ad - bc = 0$ in $\mathbb{P}^3 \cong |L|$) as $(\lambda_0\mu_0, \lambda_0\mu_1, \lambda_1\mu_0, \lambda_1\mu_1)$ for $(\lambda_0, \lambda_1), (\mu_0, \mu_1) \in \mathbb{P}^1$, then the projection map $\mathcal{C}_2 \subset |L| \times T_L(P) \to T_L(P)$ simply sends the (p-2)-dimensional vector space fiber in $\mathcal{C}_2$ over $(\lambda_0\mu_0, \lambda_0\mu_1, \lambda_1\mu_0, \lambda_1\mu_1)$ to the codimension 2 subspace of $T_L(P) \cong \mathbb{C}^p$ defined by $\mu_0 x + \mu_1 y$ and $\mu_0 y + \mu_1 z$. □

Proposition 2. *Assume that C is a connected nonhyperelliptic smooth projective curve of genus g with 2 vanishing theta nulls M_1, M_2 of the following particular form: $M_1 \cong M_0(B_1), M_2 \cong M_0(B_2)$, for effective divisors B_1, B_2 and a line bundle M_0 such that $|M_0|$ is a base point free pencil, $|M_1| = |M_0| + B_1, |M_2| = |M_0| + B_2$, and $\eta = B_1 - B_2$ is a nontrivial half-period. In other words, $|M_1|$ and $|M_2|$ are distinct vanishing theta null pencils on C with the same moving part $|M_0|$ and distinct base divisors B_1 and B_2.*

Let $\pi : \widetilde{C} \to C$ be the étale double cover of C defined by $\eta = B_1 - B_2$, and consider $L = \pi^*(M_1)$ on $\widetilde{C}$. Then $L \in \Xi$ has $h^0(\widetilde{C}, L) = 4$ and Mumford's skew symmetric matrix of linear forms can be expressed in the form displayed in Proposition 1.

Proof. Since $M_1 \cong M_2(\eta)$ on C and $\pi^*(\eta) = 0$ on $\widetilde{C}$ by construction, $\pi^*(M_1) \cong \pi^*(M_2)$. Therefore, by the projection formula, $H^0(L) \cong H^0(M_1) \oplus H^0(M_2)$ has dimension $2 + 2 = 4$. (In detail, $H^0(L) \cong H^0(\pi^*(M_1)) \cong H^0(\pi_*\pi^*(M_1)) \cong H^0(M_1 \otimes \pi_*(\mathcal{O}_{\widetilde{C}})) \cong H^0(M_1 \otimes (\mathcal{O}_C \oplus \mathcal{O}_C(\eta)) \cong H^0(M_1) \oplus H^0(M_2)$.)

Now the determination of the skew symmetric 4×4 matrix of linear forms is just like [S-V3, Ex. 2.18, p. 492] and [S-V4, Prop. 3.6, p. 246], except that everything has been pre-arranged here so that the rank of the Pfaffian quadratic form is 3, not 4. Namely, the upper right 2×2 block Λ of linear forms comes from the cup product $H^0(M_1) \times H^0(M_1(\eta)) \to H^0(\Omega_C(\eta))$. If the two base divisors B_1 and B_2, for $|M_1|$ and $|M_2| = |M_1(\eta)|$ resp., are defined by sections β_i of $\mathcal{O}_C(B_i), i = 1, 2$, then the section $\beta_1\beta_2$ of $\mathcal{O}_C(B_1 + B_2)$ is a common factor of all the entries of Λ. Dividing out $\beta_1\beta_2$, we have the 2×2 block of linear forms coming from the cup product $H^0(M_0) \times H^0(M_0) \to H^0(M_0^2)$. Now for any pencil $|M_0|$, the linear map $H^0(M_0) \otimes_{\mathbb{C}} H^0(M_0) \to H^0(M_0^2)$ has rank exactly 3 since projectively the morphism $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}H^0(M_0^2)$ given by addition of divisors in the pencil, has finite fibers. Finally, if s, t are a basis for $H^0(M_0)$ then the matrix of cup products is $\begin{pmatrix} s^2 & st \\ ts & t^2 \end{pmatrix} = \begin{pmatrix} s^2 & st \\ st & t^2 \end{pmatrix}$. Thus $\Lambda = \begin{pmatrix} s^2 & st \\ st & t^2 \end{pmatrix} \cdot \beta_1\beta_2 = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$, as desired. □

Proposition 3. *There exists a (nonhyperelliptic) trigonal connected smooth projective curve C of genus 6 with 2 vanishing theta nulls M_1, M_2 that satisfy all the properties postulated in Proposition 2.*

Proof. We will construct C as the normalization of a degree six plane curve with certain singularities and special tangent lines. First, two lemmas.

Lemma 1. Let $Y \subset \mathbb{P}^2$ be an irreducible sextic curve with a triple point P and a double point Q, both ordinary, and no other singularities. Let $C \to Y$ be the normalization. Then C is a connected smooth projective curve of genus 6 and is trigonal, nonhyperelliptic. Moreover, for any line ℓ through Q that is tangent to Y at 2 smooth points a and b, there is an associated vanishing theta null pencil on C with base divisor $a + b$.

Proof. C is a smooth projective curve by construction, since the normalization of any projective curve is a smooth projective curve. C is irreducible since Y is, hence C is connected. Now we compute the genus of C. At an ordinary plane curve singularity of multiplicity m, the "diminution of the genus" is given by the formula $\delta = \frac{m(m-1)}{2}$. Thus $\delta_P = 3$ and $\delta_Q = 1$, and hence $g(C) = 10 - (3 + 1) = 6$ (since Y has arithmetic genus $p_a(Y) = (6 - 1)(6 - 2)/2 = 10$).

The pencil of lines through P in the plane cut a moving degree 3 linear series on Y. More precisely, if p_1, p_2, p_3 are the 3 points of C mapping to P in Y (the "branches" of Y at P), then $|\mathcal{O}_C(1)(-p_1 - p_2 - p_3)|$ is a base point free pencil of degree 3 on C, where $\mathcal{O}_C(1)$ on C is the pullback of the line bundle $\mathcal{O}_Y(1)$ on Y. Since C admits a g^1_3 (degree 3 pencil), C is trigonal; and C is nonhyperelliptic since this g^1_3 is base point free (and $g > 2$).

Now assume that $\ell \subset \mathbb{P}^2$ is a line through Q that is bitangent to Y, at distinct smooth points a and b. Set $B = a + b$ on C, and let G denote the g^1_3 line bundle $\mathcal{O}_C(1)(-p_1 - p_2 - p_3)$. Then consider the degree 5 line bundle $M = G(B)$ on C. We must show that $M^{\otimes 2} \cong \Omega^1_C$, $h^0(M) = 2$, and $|M| = |G| + B$ $(= |g^1_3| + B$, the g^1_3 pencil plus base divisor $B)$.

On one hand, $M^{\otimes 2} \cong \mathcal{O}_C(2)(-2p_1 - 2p_2 - 2p_3)(2a + 2b)$, and using the line $\overline{PQ}$ to get the divisor $2(p_1 + p_2 + p_3 + q_1 + q_2 + r)$ in $|\mathcal{O}_C(2)|$, we can represent $M^{\otimes 2}$ by the divisor class $2(p_1 + p_2 + p_3 + q_1 + q_2 + r) - (2p_1 + 2p_2 + 2p_3) + (2a + 2b) = 2q_1 + 2q_2 + 2r + 2a + 2b$. On the other hand, the canonical series $|\Omega^1_C|$ is cut by the special adjoints, i.e. the plane cubics which are singular at P and contain Q, residual to the assigned base divisor $2p_1 + 2p_2 + 2p_3 + q_1 + q_2$, where q_1, q_2 are the 2 points of C mapping to Q in Y (the "branches" of Y at Q). (In particular, the degree checks: $3 \cdot 6 - 8 = 10$.) Thus take for an adjoint cubic, the line ℓ plus twice the line through P and Q. The divisor on C that this cubic curve cuts is $(q_1 + q_2 + 2a + 2b) + 2(p_1 + p_2 + p_3 + q_1 + q_2 + r)$, where r is the 6th point of intersection of $\overline{PQ}$ with Y. Subtracting the assigned base divisor gives the canonical divisor: $(2(p_1+p_2+p_3)+3(q_1+q_2)+2r+2a+2b)-(2p_1+2p_2+2p_3+q_1+q_2)=2q_1+2q_2+2r+2a+2b$. Thus, $M^{\otimes 2}\cong\Omega^1_C$.

To show the rest, since $h^0(G)=2$ and $|G|+B\subset|M|$, it will suffice to check that $h^0(M)<3$. By Clifford's theorem, $h^0(M)\leq 3$. So consider the possibility that $h^0(M)=3$. Then the net $|M|$ must be base point free since otherwise, by subtracting a base point we could get a g^2_4 and C would be hyperelliptic, contradiction. Thus $|M|$ would have to be a base point free g^2_5 and map C onto a plane quintic curve. This map would have to be an isomorphism from C to a smooth plane quintic, since a plane quintic has arithmetic genus 6 and C is smooth of genus 6. However, a smooth plane quintic curve is never trigonal (since any degree 3 effective divisor D on a smooth plane quintic imposes 3 independent linear conditions for plane conics to cut a divisor $\geq D$, so D cannot move in a linear series). □

Lemma 2. There exists an irreducible sextic curve $Y\subset\mathbb{P}^2$ with a triple point P and a double point Q, both ordinary, and no other singularities, with the property that there are 2 distinct lines through Q each of which is tangent to Y at 2 smooth points.

Proof. We will work in $\mathbb{P}^2$ with a point Q, 2 distinct lines through Q, the proposed points of bitangency, a point P not on the 2 lines, and apply Bertini's theorem to get the existence of Y. Let's begin in the affine plane with $Q=(0,0)$, the x-axis and the y-axis, points $(a_1,0),(b_1,0)$ on the x-axis and points $(0,a_2),(0,b_2)$ on the y-axis, and the point $P=(c,c)$. Now consider the affine plane sextics with the following types of equations: $A=(x\alpha_1\beta_1)^2, B=(y\alpha_2\beta_2)^2$, and $\mathcal{C}=xy\cdot\{h(x,y)\}$, where α_1 is an affine equation for the line through $(a_1,0)$ and (c,c), β_1 is an affine equation for the line through $(b_1,0)$ and (c,c), (similarly for α_2,β_2), and $\{h(x,y)\}$ are the affine quartics with at least a triple point at (c,c). Now homogenize and consider the linear system generated by A,B and $\mathcal{C}$. Then there are no base points in $\mathbb{P}^2$ outside the assigned ones $((a_1,0),(b_1,0),(0,a_2),(0,b_2),Q=(0,0)$, and $P=(c,c)$ in the affine plane), and all members Y of this linear system of sextic curves in $\mathbb{P}^2$ have the property that P is a singular point of multiplicity ≥ 3, Q is a singular point of multiplicity ≥ 2, and the 2 affine coordinate axes are tangent to Y at $(a_1,0)$ and $(b_1,0)$, $(0,a_2)$ and $(0,b_2)$. For the local conditions: that P is an ordinary triple point, that Q is an ordinary double point, and that $(a_1,0),(b_1,0),(0,a_2),(0,b_2)$ are all smooth points, it is clear that there are members satisfying these conditions (from general $xyh\in\mathcal{C}$), and hence that the general member satisfies all of these local conditions. Then Bertini does the rest: irreducible and no unassigned singularities. □

Now we complete the proof of Proposition 3. Take an irreducible plane sextic curve Y as in Lemma 2, and let C be the normalization, which is trigonal as in Lemma 1. Let M_0 be the g^1_3 line bundle (denoted by G in the proof of Lemma 1), so that $|M_0|$ is a base point free pencil on C. Then, if the 2 bitangent lines through Q are tangent to Y at smooth points a_1, b_1 and a_2, b_2 resp., set $B_1 = a_1 + b_1$ and $B_2 = a_2 + b_2$ on C, and finally let $M_i = M_0(B_i)$ for $i = 1, 2$. By Lemma 1, each $|M_i| = |M_0| + B_i$ is a vanishing theta null pencil with base divisor B_i. The only thing that remains to be checked is that the half-period $\eta = B_1 - B_2$ is nontrivial. By construction, B_1 and B_2 are distinct effective divisors of degree 2 on C. Since C is nonhyperelliptic, B_1 and B_2 cannot be linearly equivalent. □

Corollary (of Propositions 1-3). There exists a genus 6 smooth trigonal curve C with an étale double cover $\pi : \widetilde{C} \to C$ for which the theta divisor Ξ of the Prym variety has an exceptional, but not very exceptional, singular point L such that the surjective map

$$C_{|L|}(X) \to C_L(\Xi)$$

from the normal cone $C_{|L|}(X)$ of the divisor variety X along $|L|$, to the tangent cone $C_L(\Xi)$ of the Prym theta divisor at L, has the following structure. The tangent cone $C_L(\Xi)$ is irreducible and is parametrized by the part of $C_{|L|}(X)$ supported over $|L| \cap Sing(X)$, not by the unique irreducible component that dominates $|L|$. In particular, the normal cone $C_{|L|}(X)$ is reducible. □

Remark. The Prym variety appearing in the Corollary is a nonhyperelliptic genus 5 Jacobian (by results of Mumford and Recillas, see [M], [Re], and also [B1, Thm. 4.10, p. 170]), and the rank 3 double point L on the theta divisor is a vanishing theta null.

Appendix

The Riemann Singularity Theorem after Kempf and Mumford

Based on the notes "Topics on Riemann Surfaces" [K2] from a seminar presented by George Kempf at the Mathematics Institute of UNAM during August and September 1973.

In spite of the many proofs in the literature today of Riemann's singularity theorem, we believe the following notes describing the original approach of Mumford and Kempf will interest a number of readers. For one thing they contain the arguments, which were omitted from the version of Kempf's thesis published in his Annals paper [K1], that in addition to its tangent cone, the theta divisor itself has local determinantal equations. Secondly, the notes of Kempf's unpublished seminar which these very closely follow, are written in relatively elementary language. The present authors feel that the resulting exposition by Kempf is a candidate for the clearest treatment of Riemann's singularity theorem available and we want to take this opportunity to publicize it more widely. Although we have not been able to preserve the elegant brevity of Kempf's original words, we hope the clarity of the argument is not entirely lost. The complete arguments were published by Kempf, in a more abstract form, in the monograph [K3], now unfortunately out of print.

A primary tool for studying a compact Riemann surface C of genus g, is the family of Abel maps $\alpha : C^{(i)} \to Pic^i(C)$, where $C^{(i)}$ is the ith symmetric power of C and $Pic^i(C) \cong J = H^0(C;K)^*/H_1(C;\mathbb{Z})$ is its Picard or Jacobian variety. Recall the map $\alpha : C^{(i)} \to Pic^i(C)$ takes a divisor D to the line bundle $\mathcal{O}(D)$. As a map to $H^0(C;K)^*/H_1(C;\mathbb{Z})$, α sends D to the linear function on one-forms defined by integration from D_0 to D, where D_0 is a fixed divisor of degree i. As an example of its importance, the Riemann Roch problem for D, is equivalent to computing the rank of the derivative of α at D.

In particular, a fundamental problem is to relate analytic properties of C to geometric invariants of the images $\alpha(C^{(i)}) = W^i =$ the subvariety in $Pic^i(C)$ of effective line bundles of degree i on C. E.g. for $i \leq g-1$, one can show L is a smooth point of W^i if and only if $h^0(C;L) = 1$. The Riemann singularities theorem makes this even more precise when $i = g-1$.

For the case $i = g-1$, Riemann gave a holomorphic function ϑ defining the inverse image in $H^0(C;\mathbb{C})^* \cong$ complex g space, of $W^{g-1} =$ the "theta divisor" of C, and used it to give an extremely useful intrinsic formula for the multiplicity of W^{g-1} at a point L representing a line bundle of degree $g-1$ on C. Namely,

Theorem (Riemann). $mult_L(W^{g-1}) = h^0(C;L) = 1 + dim(\alpha^{-1}(L))$

Remark. A local equation for W^{g-1} near L is determined only up to multiplication by a unit in the analytic local ring of J at L, but its leading term, the equation of the tangent cone of W^{g-1} at L, is determined up to multiplication by a non zero constant. Hence not only the order, but also the homogeneous polynomial itself of lowest degree in a local equation for W^{g-1} near L has intrinsic significance, at least as an element of the symmetric algebra on $H^0(C;K)$. Kempf improved Riemann's numerical result by giving an intrinsic construction for this homogeneous polynomial of degree $h^0(L)$, defining the tangent cone of W^{g-1} at L, as follows.

Theorem (Kempf). *Algebraically, the tangent cone in T_LJ to W^{g-1} at a point L representing a line bundle of degree $g-1$ on C, is defined by an irreducible polynomial of the form $det[X_iY_j]$, where $\{X_i\}$ is a basis of $H^0(C;L)$ and $\{Y_j\}$ is a basis of $H^0(C;K-L)$. In particular it has degree $h^0(C;L)$.*

Geometrically, the projectivized tangent cone is a reduced irreducible rational variety with support equal to the union of the spans of the divisors D in $|L|$ in the canonical space PT_LJ of the curve C, and is parametrized birationally by the projective normal bundle to the fiber $\alpha^{-1}(L) = |L|$ in $C^{(g-1)}$ of the Abel map via the derivative α_* of that map.

Remarks. 1) Kempf generalized his result to describe the tangent cones to all W^i, for $i \leq g-1$. In this note we explain first the easier case of W^{g-1}, and indicate the generalization for $i < g-1$ at the end.

2) In particular, the theorem gives the same equation for the tangent cones to W^{g-1} at L and at $K-L$, as expected.

3) In the case $h^0(C;L) = 2$, Andreotti and Mayer had previously given the equation of the quadratic tangent cone to W^{g-1}.

Indeed Kempf once remarked to us that he discovered his result after reading their paper, which suggests he understood Lemma 4 pp. 192-193, and Prop. 8c) pp. 210-211 of [A-M], and their proofs, very well indeed.

With hindsight the discussion in [A-M], Prop. 8c), p. 211, and lemma 4, p. 193, can be summarized as follows. Given a line bundle L on C of degree $g-1$ with $h^0(C;L) = 2$, the multiplicity of W^{g-1} at L is 2, and the homogeneous quadratic Taylor polynomial for ϑ at L vanishes precisely on the spans of all the divisors D in the linear series $|L|$, [Prop. 8c), proof: p. 211].

Moreover let $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ be bases of $H^0(C;L)$ and $H^0(C;K-L)$, defining divisors D_1, D_2 in $|L|$ and E_1, E_2 in $|K-L|$. Then X_iY_j is a one-form on C representing a homogeneous linear equation for the hyperplane cutting the divisor $D_i + E_j$ on the canonical model of C, and $det[X_iY_j] = X_1Y_1 \cdot X_2Y_2 - X_1Y_2 \cdot X_2Y_1$, is an equation for the quadric tangent cone Q to W^{g-1} at L, [A-M, lemma 4, proof: p. 193].

Moreover the zero locus of this determinant is doubly ruled, once by the common zeroes of the entries in each linear combination of the rows, i.e. by the spans of the divisors in $|L|$, and once by the common zeroes of the entries in each linear combination of the columns, i.e. by the spans of the divisors in $|K-L|$. In the case examined in [A-M] these are the ruling pencils of a quadric of rank 3 or 4, given in line -5, p. 191, (with a misprint).

Proof of Kempf's theorem. By definition the tangent cone to W^{g-1} at L is the leading term of a local equation for W^{g-1} near L, so the first step is to find such an equation. Since the Abel map fibers the symmetric products $C^{(i)}$ over J with linear projective space fibers, one can add a base divisor so that all fibers are subspaces of projective spaces of the same dimension, hence locally, in both the analytic and the algebraic Zariski topologies, all fibers become subspaces of the same projective space. Then the idea, attributed by Kempf to Mumford, is to use the most elementary case of elimination theory to give a determinantal equation for the image of $C^{(g-1)}$ locally near L.

Recall $\alpha : C^{(2g)} \to Pic^{2g}(C)$ is a locally trivial family of projective spaces $\alpha^{-1}(L) \cong \mathbb{P}^g$, for all L in $Pic^{2g}(C)$. Moreover, if $E = P_0 + ... + P_g$ is a divisor consisting of $g+1$ distinct points P_i, then adding E to each effective divisor of degree $g-1$, defines an embedding $+E : C^{(g-1)} \hookrightarrow C^{(2g)}$.

Moreover the image $C^{(g-1)} + E = \{D \in C^{(2g)} : D \geq E\}$ is the transverse intersection of the $g+1$ smooth subvarieties $C^{(2g-1)} + P_i$. Thus for each L in $Pic^{g-1}(C)$, the Abel fiber $|L|$ in $C^{(g-1)}$ is isomorphic to the subspace $|L| + E$ of the Abel fiber $|L \otimes \mathcal{O}(E)| \cong \mathbb{P}^g$ in $C^{(2g)}$. Moreover $|L| + E$ is the subspace of $|L \otimes \mathcal{O}(E)|$ defined as the common zeroes of the $g+1$ linear equations obtained by setting sections of $L \otimes \mathcal{O}(E)$ equal to zero at each P_i. At a general L, the $g+1$ hyperplanes of $|L \otimes \mathcal{O}(E)| \cong \mathbb{P}^g$ defined by the points P_i are independent and $|L| =$ empty set. Indeed these hyperplanes are dependent precisely when L belongs to W^{g-1}.

Thus locally over a neighborhood U of $L \otimes \mathcal{O}(E)$ in J, the smooth variety $C^{(2g)}$ is analytically isomorphic to $U \times \mathbb{P}^g$, and the smooth subvariety $C^{(g-1)} + E \cong C^{(g-1)}$ of $C^{(2g)}$, is defined locally in $U \times \mathbb{P}^g$ by $g+1$ transverse simultaneous equations $\sum_j a_{j,k}(u) X_j = 0$, for $0 \leq k \leq g$. Here the $a_{j,k}$ are analytic functions on U and $X_0, ..., X_g$ are homogeneous linear coordinates on $\mathbb{P}^g$. For each u in U, there is a point of $C^{(g-1)} + E$ lying over u if and only if the matrix $[a_{j,k}(u)]$ is singular. Then in these local coordinates the image $W^{g-1} + E \cong W^{g-1}$, of $C^{(g-1)} + E \cong C^{(g-1)}$, is defined as a set near $L \otimes \mathcal{O}(E)$, by one equation $det[a_{j,k}(u)] = 0$. Hence, after translation, the same holds for the variety W^{g-1} near L, in $Pic^{g-1}(C)$.

Next one can use this local equation to compute the degree and some geometry of the tangent cone to W^{g-1} at L, but first we must show this equation defines W^{g-1} with the correct, i.e. reduced, scheme structure. Since W^{g-1} is the image of a smooth irreducible variety by a map with connected fibers, W^{g-1} is globally irreducible as well as locally analytically irreducible near every point L. Thus to show the determinant equation is reduced, it suffices to check that the scheme structure defined by this equation is non singular at some point. To give more control over these equations we begin by putting the matrix in simpler form adapted to the dimension of the fiber.

By Abel's theorem we know the projective linear fiber $|L|$ has dimension $r = h^0(L) - 1 \geq 0$. Hence the corank of the numerical matrix $[a_{j,k}(L)]$ equals $r+1 = h^0(C; L)$. Now apply row and column operations in the analytic local ring of Pic^{g-1} near L. Then $[a_{j,k}]$ reduces to a matrix of analytic functions with $(g-r)$ ones on the upper left diagonal, all zeroes in the upper right and lower left blocks, and a square matrix $[b_{j,k}]$ of dimension $h^0(C; L)$ in the lower right hand corner, with $0 \leq j, k \leq r$. Moreover all the functions $b_{j,k}$ vanish at L, and the determinant has at most been multiplied by a unit in the local ring.

Then $C^{(g-1)}$ is defined locally near the fiber $|L| \cong \{L\} \times \mathbb{P}^r$, in $U \times \mathbb{P}^r$, by the system of $r+1 = h^0(C; L)$ equations: $\sum_j b_{j,k}(u) X_j = 0$, for $0 \leq k \leq r$, where $X_0, ..., X_r$ are homogeneous linear coordinates on $\mathbb{P}^r$. Consequently W^{g-1} is defined as a set near L in U by one equation $det[b_{j,k}] = 0$, where $[b_{j,k}]$ is an $h^0(C; L)$ by $h^0(C; L)$ matrix of analytic functions on U, all vanishing at L. Since $C^{(g-1)} \to W^{g-1}$ is surjective, $dim(W^{g-1}) \leq dim(C^{(g-1)}) = g-1$.

Since W^{g-1} is defined by one equation, $dim(W^{g-1}) \geq g-1$ as well. Thus $dim(W^{g-1}) = g-1$, and the general fiber $|M|$ of the map $C^{(g-1)} \to W^{g-1}$ has dimension zero, thus is a single point.

To show the local equation $det[b_{j,k}] = 0$ defines W^{g-1} with its reduced structure, we will show the structure it defines for W^{g-1} is isomorphic to $C^{(g-1)}$ locally over a general point M of W^{g-1} in U. Choose a point M in U with $h^0(C; M) = 1$. Then near M, the diagonalization procedure yields an equation for W^{g-1} which is a unit multiple of the previous one, and has form $det[c_{0,0}(u)] = c_{0,0}(u)$, where $C^{(g-1)}$ is defined near $|M| = \{M\}$ by one equation $c_{0,0}(u) \cdot X_0$, in $U \times \mathbb{P}^0$. Thus near M, our equation for W^{g-1} is a unit times the equation for $C^{(g-1)}$ near $|M| = \{M\}$, so W^{g-1} is indeed isomorphic to the smooth variety $C^{(g-1)}$ locally over M.

Since the equation $det[b_{j,k}]$ defines W^{g-1} as a reduced scheme near L, we can use it to compute the tangent cone by finding its leading term. Since the constant term of every $b_{j,k}$ is zero, all Taylor polynomials of $det[b_{j,k}]$ of degree $< h^0(C; L)$ are zero, and the first possibly non zero term is $det[B_{j,k}]$ where $B_{j,k}$ is the linear term of $b_{j,k}$. The easy direction of Riemann's theorem, that $mult_L(W^{g-1} \geq h^0(C; L)$, follows from this fact.

To deduce the other direction, that $mult_L(W^{g-1}) \leq h^0(C; L)$, it suffices to show the polynomial $det[B_{j,k}]$ is not identically zero. Since the terms $B_{j,k}$ are linear functions on $\mathbb{P}T_L J$, $det[B_{j,k}]$ is a homogeneous polynomial of degree $h^0(C; L)$ on $\mathbb{P}T_L J$. To show it is not zero, we claim it vanishes only on a subset of codimension one in $\mathbb{P}T_L J \cong \mathbb{P}^{g-1}$, namely on the image under the differential of the Abel map, of the $(g-2)$-dimensional projective normal bundle to the fiber $|L|$ in $C^{(g-1)}$.

To see that, note that to compute that normal bundle, we compute the tangent bundle to $C^{(g-1)}$ along $|L|$ and take the part normal to $|L|$. Since $C^{(g-1)}$ is defined by the equations $\sum_j b_{j,k} \cdot X_j = 0$, $0 \leq k \leq r$, in $U \times \mathbb{P}^r$, with $|L| \cong \{L\} \times \mathbb{P}^r$, the product rule for derivatives implies this normal bundle is the subset of $\mathbb{P}T_L J \times \mathbb{P}^r$ defined by the $r+1$ equations $\sum_j B_{j,k} \cdot X_j = 0$, $0 \leq k \leq r$, where $X_0, ..., X_r$ are homogeneous coordinates on $\mathbb{P}^r$.

Hence by the same argument given above for the zero locus of $det[b_{j,k}]$, $det[B_{j,k}]$ vanishes precisely on the image in $\mathbb{P}T_L J$ of the projectivized normal bundle to $|L|$ in $C^{(g-1)}$. Since that projective bundle has dimension $g-2$, its image in $\mathbb{P}T_L J \cong \mathbb{P}^{g-1}$ is a proper subvariety, and thus $det[B_{j,k}]$ is not identically zero. □

Corollary. The tangent cone to W^{g-1} at a point L, is a reduced irreducible rational variety of degree $h^0(C; L)$, whose support is the image of the normal cone to the fiber $|L|$ in $C^{(g-1)}$, via the derivative of the Abel map.

Proof. The fact that the normal cone is irreducible implies the same for its image the tangent cone. Next, the map from the rational normal cone to the tangent cone is birational by the same argument as given above for the map $C^{(g-1)} \to W^{g-1}$. Hence the tangent cone is also rational and reduced. □

The ideas in this first part of the proof are attributed by Kempf to Mumford, [K2] and [K3, p. 160]. This already gives a complete proof of the classical Riemann singularity theorem, on the multiplicity of W^{g-1}, i.e. the degree of the tangent cone.

To complete the proof of Kempf's theorem, next we want to make explicit the set theoretic support of the projective tangent cone to W^{g-1} at L, and give Kempf's intrinsic equation for that tangent cone.

Lemma. For any $d \geq 1$, any line bundle L of degree d on C, and any divisor D in $|L|$, the image under the derivative of the Abel map, of the projective normal space to $|L|$ in $C^{(d)}$ at D, is the span of the divisor D on the canonical model of C in $\mathbb{P}T_LJ \cong \mathbb{P}H^0(C;K)^*$.

Proof. For a divisor D of distinct points, this follows from the fundamental theorem of calculus applied to the entries in the abelian integral form of the Abel map. It is implied for general D, (as well as the dimension of the span of D), by the exactness of the sequence on page 30 of Mattuck and Mayer [M-M]. The exactness is proved there, and is proved to be equivalent to the Riemann Roch theorem for D. □

Corollary. The image under the derivative of the Abel map, of the projective normal bundle to $|L|$ in $C^{(d)}$, is the union of the spans of the divisors D in $|L|$, on the canonical model of C in $\mathbb{P}T_LJ$.

The algebraic part of Kempf's theorem is contained in the following proposition.

Proposition. *The equation of the tangent cone to W^{g-1} at L is given by $det[X_iY_j] = 0$, where $\{X_i\}$ is a basis for $H^0(C;L)$ and $\{Y_j\}$ is a basis for $H^0(C;K-L)$. The polynomial $det[X_iY_j]$ is irreducible.*

Proof. If non zero, this polynomial has the same degree as the reduced and irreducible tangent cone, so if it also vanishes on that cone it must be irreducible. If D_i in $|L|$ is the divisor of $X_i = 0$, then the entries of the ith row of the matrix $[X_iY_j]$ form a basis for the one forms vanishing on D_i, hence as linear functions on canonical space $\mathbb{P}T_LJ$ they cut out the span of D_i. In particular that row becomes zero in the numerical matrix obtained by evaluating $[X_iY_j]$ at any point p of the span of D_i, so the matrix $[X_iY_j(p)]$ is singular for any p in that span. Similarly if D is any divisor of $|L|$, i.e. the divisor of any linear combination of the X_i, then that same linear combination of the rows of $[X_iY_j]$ becomes zero at any point p of the span of D. Thus $[X_iY_j(p)]$ is singular if p belongs to the span of any divisor D in $|L|$.

Since the polynomial $det[X_iY_j]$ vanishes on the reduced and irreducible tangent cone of degree $h^0(C;L)$ to W^{g-1} at L, the polynomial must be either identically zero, or precisely the equation of degree $h^0(C;L)$ of that cone. To see it is not identically zero, let p be a point where it vanishes. Then some linear combination of the rows of $[X_iY_j(p)]$ is zero. Then p belongs to the common zero locus of the entries in that vector of one forms, which are a basis for the one forms vanishing on the zero divisor D of the corresponding linear combination of the X_i. Thus p does lie on the span of some divisor D of $|L|$. Since the union of these spans is a hypersurface in canonical space, $det[X_iY_j]$ does not vanish identically. □

Remarks. We want to outline Kempf's achievement of generalizing these results to all W^i with $i \leq g-1$.

Theorem (Kempf). *Consider W^i with its reduced scheme structure. Algebraically, the tangent cone in T_LJ to W^i at a point L representing an effective line bundle of degree i on C, is defined by a prime ideal generated by the maximal minor determinants of the matrix $[X_jY_k]$, where $\{X_j\}$ is a basis of $H^0(C;L)$ and $\{Y_k\}$ is a basis of $H^0(C;K-L)$. The multiplicity of W^i at L equals the binomial coefficient $\binom{h^1(L)}{dim|L|}$.*

Geometrically, the projectivized tangent cone to W^i at L is a reduced irreducible rational variety with support equal to the union of the spans of the divisors D in $|L|$ in the canonical space $\mathbb{P}T_LJ$ of the curve C, and is parametrized birationally by the projective normal bundle to the fiber $\alpha^{-1}(L) = |L|$ in $C^{(g-1)}$, of the Abel map.

Proof (sketch). When $i < g-1$, the technique of adding a base divisor and performing row and column operations yields a matrix, no longer square, of local analytic functions whose maximal minors vanish exactly on the set theoretic image W^i of the Abel map on $C^{(i)}$. An argument similar to the case $i = g-1$ shows the scheme structure defined on W^i by these determinants is again isomorphic to the smooth scheme structure of $C^{(i)}$ near a point L of W^i over which the fiber of the Abel map is a single point. It follows as before that this determinantal scheme structure on W^i is generically reduced, but since W^i is not a hypersurface when $i < g-1$, it is no longer clear it is actually reduced.

At this point Kempf invokes "unmixedness" results for determinantal varieties, which he attributes originally to Macaulay and which he also reproves, to deduce that the given determinants do define W^i with its reduced scheme structure. It follows that the tangent cone of W^i is defined by the ideal of homogeneous leading terms, not however of just the generators of the ideal for W^i, but of every element of that ideal. Kempf uses Macaulay's results to overcome that problem as well and show that it suffices to consider only the leading terms of the given determinantal generators as follows.

First he considers, as in the case $i = g-1$, the matrices whose entries are the linear terms of the entries in the matrices defining W^i. The maximal minors of these matrices give some of the initial homogeneous forms belonging to the ideal defining the scheme structure of the tangent cone to

W^i at L. Moreover, as before, these maximal minors of matrices of linear forms define the image of the projective normal bundle to $|L|$ in W^i, under the derivative of the Abel map, which by the universal property of blowing up equals the tangent cone to W^i, at least set theoretically.

Since these minors are only a special subset of the homogeneous generators of the ideal of the tangent cone, it is possible that the actual tangent cone is a closed subscheme of the scheme structure they define. Repeating the arguments for the determinantal equations of W^i, it follows again that the scheme structure on T_LW^i defined by the maximal minors is generically reduced. Invoking the unmixedness results again implies that these minors define the reduced scheme structure on T_LW^i. Since this is the smallest possible scheme structure on T_LW^i and the tangent cone is a subscheme of this one, this is indeed the actual tangent scheme structure. Thus the tangent cone to W^i is reduced, and is the birational image of the irreducible and rational projective normal bundle to $|L|$ in $C^{(i)}$, via the derivative of the Abel map.

To give explicit equations for the tangent cone at any line bundle L of degree $i < g-1$, Kempf again considers bases $\{X_j\}$ of $H^0(C;L)$, and $\{Y_k\}$ of $H^0(C;K-L)$. Let $\{T_j\}$ be another basis of $H^0(C;L)$. Then the theorem of Mattuck and Mayer implies the projective normal bundle of $|L|$ in $C^{(i)}$ maps by the derivative of the Abel map, and bundle projection, isomorphically to the smooth subscheme of $\mathbb{P}T_LJ \times |L|$ defined by the equations $\{\sum_j (X_jY_k) \cdot T_j = 0, \text{all } k\}$.

To show the common zero scheme in $\mathbb{P}T_LJ \times |L|$ of the equations $\{f_k = \sum_j (X_jY_k)T_j = 0$ for all $k\}$ is smooth, we calculate the differentials on the product of the associated vector spaces. Each function $f_k = \sum_j (X_jY_k)T_j$ is bilinear on the product $T_LJ \times H^0(L)$, hence linear in each factor. Thus the partial differential in each factor direction equals that same linear function. In particular, at a point (a, b) in the product, the partial differential of f_k in the T_LJ direction is the linear function $L_k = \sum_j (X_jY_k)b_j = (\sum_j b_jX_j) \cdot Y_k$. In particular it is independent of a. These linear functions L_k are independent in $H^0(C;K)$ because the sections Y_k are independent in $H^0(C;K-L)$.

Consequently, the tangent cone to W^i at L is the image of the projection of this subscheme into $\mathbb{P}T_LJ$, i.e. the set of points where all $h^0(C;L)$ by $h^0(C;L)$ subdeterminants of the matrix $[X_jY_k]$ vanish. The same proof of Kempf's proposition above implies this locus, the set of points p at which the matrix $[X_jY_k(p)]$ has dependent rows, equals the union of the spans in canonical space, of the divisors D in $|L|$.

Arguing as above and using again the unmixedness results for determinantal varieties, the scheme structure defined on the tangent cone by this explicit matrix of linear forms is reduced, hence these determinants define the tangent cone as a scheme. Since this is the reduced scheme structure on an irreducible variety, these determinants generate a prime ideal.

Finally Kempf computes the degree of the tangent cone by computing its cycle class as a push forward of the cycle class of the normal bundle in $\mathbb{P}T_LJ \times |L|$ defined by the equations $\{\sum_j (X_jY_k) \cdot T_j = 0, \text{all } k\}$. I.e. these $h^0(C; K-L)$ equations of type $(1,1)$ define the projective normal bundle N as a complete intersection. If h_1 and h_2 are hyperplanes in $\mathbb{P}T_LJ$ and $|L|$ respectively, the cycle class of the zero locus of a single $(1,1)$ form is the class of $(h_1 \times |L|) + (\mathbb{P}T_LJ \times h_2)$.

Hence the class of N is the $h^0(C; K-L)$-fold self intersection of this class, which by the binomial theorem equals:

$$\sum_{\substack{n+m=h^0(C;K-L)\\ n,m\geq 0}} \frac{(n+m)!}{n!\,m!}\, h_1^n \cdot h_2^m$$

where h^s is the class of the intersection of s general hyperplanes in the projective space containing h, h^0 equals that whole projective space, and $h_1^n \cdot h_2^m$ is the class of the Cartesian product of those intersections, in the product of both projective spaces.

Projecting into $\mathbb{P}T_LJ$, the only non zero, i.e. dimension preserving, projection of this class occurs when h_2^m is zero dimensional, i.e. the component with $m = dim|L|$, which gives $\dfrac{(h^0(C;K-L))!}{(dim|L|)!\,(g-i)!}\, h_1^{g-i}$. Hence the multiplicity of W^i at L equals $\binom{h^1(L)}{dim|L|}$. □

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