Math 8320 Spring 2004, Riemann's view of plane curves

Riemann's goal was to classify all complex holomorphic functions of one variable.

1) The fundamental equivalence relation on power series: Consider a convergent power series as representing a holomorphic function in an open disc, and consider two power series as representing the same function if one is an analytic continuation of the other.

2) The monodromy problem: Two power series may be analytic continuations of each other and yet not determine the same function on the same open disc in the complex plane, so a family of such power series does not actually define a function.

Riemann's solution: Construct the ("Riemann") surface S on which they do give a well defined holomorphic function, by considering all pairs (U,f) where U is an open disc, f is a convergent power series in U, and f is an analytic continuation of some fixed power series f_0. Then take the disjoint union of all the discs U, subject to the identification that on their overlaps the discs are identified if and only if the (overlap is non empty and the) functions they define agree there.

Then S is a connected real 2 dimensional manifold, with a holomorphic structure and a holomorphic projection S-->C mapping S to the union (not disjoint union) of the discs U, and f is a well defined holomorphic function on S.

3) Completing the Riemann surface: If we include points where f is meromorphic, and allow discs U which are open neighborhoods of the point at infinity on the complex line, we get a holomorphic projection S-->P^1 = C union {p}, and f also defines a holomorphic function S-->P^1.

4) Classifying functions by means of their Riemann surfaces:
This poses a new 2 part problem:
(i) Classify all the holomorphic surfaces S.
(ii) Given a surface S, classify all the meromorphic functions on S.

5) The fundamental example
Given a polynomial F(z,w) of two complex variables, for each solution pair F(a,b) = 0, such that $\partial F/\partial w$ (a,b) $\neq$ 0, there is by the implicit function theorem, a neighborhood U of a, and a nbhd V of b, and a holomorphic function w = f(z) defined in U such that for all z in U, we have f(z) = w if and only if w is in V and F(z,w) = 0. I.e. we say F determines w = f(z) as an "implicit" function. If F is irreducible, then any two different implicit functions determined by F are analytic continuations of each other. For instance if F(z,w) = z-w^2, then there are for each a $\neq$ 0, two holomorphic functions w(z) defined near a, the two square roots of z.

In this example, the surface S determined by F is essentially equal to the closure of the plane curve X: {F(z,w) = 0}, in the projective plane P^2. More precisely, S is constructed by removing and then adding back a finite number of points to X as follows.

Consider the open set of X where either $\partial F/\partial w\,(a,b) \neq 0$ or $\partial F/\partial z\,(a,b) \neq 0$. These are the non singular points of X. To these we wish to add some points in place of the singular points of X. I.e. the set of non singular points is a non compact manifold and we wish to compactify it.

Consider an omitted i.e. a singular point p of X. These are always isolated, and projection of X onto an axis, either the z or w axis, is in the neighborhood of p, a finite covering space of the punctured disc U^* centered at the z or w coordinate of p. All such connected covering spaces are of form $t \dashrightarrow t^r$ for some $r \geq 1$, and hence the domain of the covering map, which need not be connected, is a finite disjoint union of copies of U^*. Then we can enlarge this space by simply adding in a separate center for each disc, making a larger 2 manifold.

Doing this on an open cover of X in P^2, by copies of the plane C^2, we eventually get the surface S, which is in fact compact, and comes equipped with a holomorphic map S-->X, which is an isomorphism over the non singular points of X. S is thus a "desingularization" of X. For example if X crosses itself with two transverse branches at p, then S has two points lying over p, one for each branch or direction. If X has a cusp, or pinch point at p, but a punctured neighborhood of p is still connected, there is only one point of S over p, but it is not pinched.

Theorem: (i) The Riemann surface S constructed from an irreducible polynomial F is compact and connected, and conversely, every compact connected Riemann surface arises in this way.
(ii) The field of meromorphic functions M(S) on S is isomorphic to the field of rational functions k(C) on the plane curve C, i.e. to the field generated by the rational functions z and w on C.

Corollary: The study of compact Riemann surfaces and meromorphic functions on them is equivalent to the study of algebraic plane curves and rational functions on them.

6) Analyzing the meromorphic function field M(S).
If S is any compact R.S. then M(S) = C(f,g) is a finitely generated field extension of C of transcendence degree one, hence by the primitive element theorem, can be generated by two elements, and any two such elements define a holomorphic map S-->X in $P^{\wedge 2}$ of degree one onto an irreducible plane algebraic curve, such that k(X) = M(S).

Question: (i) Is it possible to embed S isomorphically onto an algebraic curve, either one in $P^{\wedge 2}$ or in some larger space $P^{\wedge n}$?
(ii) More generally, try to classify all holomorphic mappings S-->$P^{\wedge n}$ and decide which ones are embeddings.

Riemann's intrinsic approach:
Given a holomorphic map f:S-->P^n, with homogeneous coordinates $z_0,\ldots,z_n$ on $P^{\wedge n}$, the fractions z_i/z_0 pull back to meromorphic functions $f_1,\ldots,f_n$ on S, which are holomorphic on S_0 =

$f^{-1}(z_0 \neq 0)$, and these f_i determine back the map f. Indeed the f_i determine the holomorphic map $S_0 \to C^n = \{z_0 \neq 0\}$ in P^n.

Analyzing *f* by the poles of the f_i

Note that since the f_i are holomorphic in $f^{-1}(z_0 \neq 0)$, their poles are contained in the finite set $f^{-1}(z_0 = 0)$, and on that set the pole order cannot exceed the order of the zeroes of the function z_0 at these points. I.e. the hyperplane divisor $\{z_0 = 0\}:H_0$ in P^n pulls back to a "divisor" $\sum n_j p_j$ on S, and if $f_i = z_i/z_0$ then the meromorphic function f_i has divisor $\mathrm{div}(f_i) = \mathrm{div}(z_i/z_0) = \mathrm{div}(z_i) - \mathrm{div}(z_0) = f^*(H_i) - f^*(H_0)$. Hence $\mathrm{div}(f_i) + f^*(H_0) = f^*(H_i) \geq 0$, and this is also true for every linear combination of these functions.

I.e. the pole divisor of every f_i is dominated by $f^*(H_0) = D_0$. We give a name to these functions whose pole divisor is dominated by D_0.

Definition: $L(D_0) = \{f \text{ in } M(S): f = 0 \text{ or } \mathrm{div}(f) + D_0 \geq 0\}$.

Thus we see that a holomorphic map f:S-->P^n is determined by a subspace of $L(D_0)$ where $D_0 = f^*(H_0)$ is the divisor of the hyperplane section H_0.

Theorem(Riemann): For any divisor D on S, the space L(D) is finite dimensional over C. Moreover, if g = genus(S) as a topological surface,
(i) $\deg(D) + 1 \geq \dim L(D) \geq \deg(D) + 1 - g$.
(ii) If there is a positive divisor D with $\dim L(D) = \deg(D)+1$, then $S \approx$ P^1.
(iii) If $\deg(D) > 2g-2$, then $\dim L(D) = \deg(D)+1-g$.

Corollary of (i): If $\deg(D) \geq g$ then $\dim(L(D)) \geq 1$, and $\deg(D) \geq g+1$ implies $\dim L(D) \geq 2$, hence, there always exists a holomorphic branched cover S-->P^1 of degree $\leq g+1$.

Q: When does there exist such a cover of lower degree?

Definition: S is called hyperelliptic if there is such a cover of degree 2, if and only if M(S) is a quadratic extension of C(z).

Corollary of (iii): If $\deg(D) \geq 2g+1$, then L(D) defines an embedding S-->P^(d-g), in particular S always embeds in P^(g+1).

In fact S always embeds in P^3.
Question: Which S embed in P^2?

Remark: The stronger Riemann Roch theorem implies that if K is the divisor of zeroes of a holomorphic differential on S, then L(K) defines an embedding in P^(g-1), the "canonical embedding", if and only if S is not hyperelliptic.

7) Classifying projective mappings

To classify all algebraic curves with Riemann surface S, we need to classify all holomorphic mappings S-->X in P^n to curves in projective space. We have associated to each map f:S-->P^n a divisor D_0 that determines f, but the association is not a natural one, being an arbitrary choice of the hyperplane section by H_0. We want to consider all hyperplane sections and ask what they have in common. If h: $\sum c_j z^j$ is any linear polynomial defining a hyperplane H, then h/z_0 is a rational function f with div(f) = $f^*(H)-f^*(H_0) = D-D_0$.

Definition: Two divisors D,D_0 on S are linearly equivalent and write $D \approx D_0$, if and only if there is a meromorphic function f on S with $D-D_0$ = div(f), iff D = div(f)+D_0.

In particular, $D \approx D_0$ implies that L(D) isom. L(D_0) via multiplication by f. and L(D) defines an embedding iff L(D_0) does so. Indeed from the isomorphism taking g to fg, we see that a basis in one space corresponds to a basis of the other defining the same map to P^n, i.e. ($f_0,\ldots,f_n$) and ($ff_0,\ldots,ff_n$) define the same map.

Thus to classify projective mappings of S, it suffices to classify divisors on S up to linear equivalence.

Definition: Pic(S) = set of linear equivalence classes of divisors on S.

Fact: The divisor of a meromorphic function on S has degree zero.

Corollary: Pic(S) = $\sum$ $Pic^d(S)$ where d is the degree of the divisors classes in $Pic^d(S)$.

Definition: $Pic^0(S)$ = Jac(S) is called the Jacobian variety of S.

Definition: $S^{(d)}$ = (Sx..xS)/Sym_d = dth symmetric product of S
= set of positive divisors of degree d on S.

Then there is a natural map $S^{(d)}$-->$Pic^d(S)$, taking a positive divisor D to its linear equivalence class O(D), called the Abel map. [Actually the notation O(D) usually denotes another equivalent notion: the locally free rank one sheaf determined by D.]

Remark: If L is a point of $Pic^d(S)$ with d > 0, L = O(D) for some D>0 if and only if dimL(D) > 0.
Proof: If D > 0, then C is contained in L(D). And if dimL(D)>0, then there is an f $\neq$ 0 in L(D) hence D+div(f) $\geq$ 0, hence > 0.**QED.**

Corollary: The map $S^{(g)}$-->Pic^g (S) is surjective.

Proof: Riemann's theorem showed that dimL(D)>0 if deg(D) $\geq$ g. **QED.**

It can be shown that Pic^^{g} hence every Pic^^{d} can be given the structure of algebraic variety of dimension g. In fact.

Theorem: (i) Pic^^{d}(S) isom C^^{g}/L, where L is a rank 2g lattice subgroup of C^^{g}.

(ii) The image of the map S^$^{(g-1)}$-->Pic^$^{(g-1)}$(S) is a subvariety "theta" of codimension one, i.e. dimension g-1, called the "theta divisor".

(iii) There is an embedding Pic^$^{(g-1)}$-->P^^{N} such that 3.theta is a hyperplane section divisor.

(iv) If O(D) = L in Pic^$^{(g-1)}$(S) is any point, then dimL(D) = mult$_{L}$(theta).

(v) If g(S) $\geq$ 4, then g-3 $\geq$ dim(sing(theta)) $\geq$ g-4, and dim(sing(theta)) = g-3 iff S is hyperelliptic.

(vi) If g(S) $\geq$ 5 and S is not hyperelliptic, then rank 4 double points are dense in sing(theta), and the intersection in P(T$_{0}$Pic^$^{(g-1)}$(S)) isom P^$^{(g-1)}$, of the quadric tangent cones to theta at all such points, equals the canonically embedded model of S.

(vii) Given g,r,d $\geq$0, every S of genus g has a divisor D of degree d with dimL(D) $\geq$ r+1 iff g-(r+1)(g-d+r) $\geq$ 0.

Next we discuss how to clasify all Riemann surfaces of genus g, using the idea of a moduli space. (to be continued?)