# MATH 3300 - EXAM 2 PRACTICE

Name: Solutions

- No notes, calculators, other electronic devices are allowed.
- Full credit may not be given if sufficient justification is not provided.
- Academic Honesty Student Honor Code: "I will be academically honest in all of my academic work and will not tolerate academic dishonesty of others."
- This exam has 9 pages. You must check that no page is missing.

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**Question 1.** (10 points) Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$.

1. (3 points) Find the reduced echelon form (columns with leading 1s must have 0s everywhere else) of $A$. Show all your work.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. (3 points) Find a basis for the column space of $A$.

Leading 1s in columns 1, 2 so $\left\{ \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} \right\}$.

3. (3 points) Find a basis for the row space of $A$.

Leading 1s in rows 1, 2 so

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

4. (1 point) Find dim(Nul($A$)).

$\dim(\text{Nul}(A)) = \underset{\#\text{ columns}}{4} - \dim(\text{Col}(A)) = 4 - 2 = 2$.

**Question 2.** (10 points) Let $A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 5 & 0 & 7 \\ 6 & 7 & 1 & 9 \\ 1 & -1 & 0 & 2 \end{bmatrix}$.

1. (5 points) Find det($A$). Show all your work.

Expanding along the 3rd column

$$\det(A) = + \begin{vmatrix} 1 & 2 & 4 \\ 0 & 5 & 7 \\ 1 & -1 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 5 & 7 \\ -1 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 4 \\ 5 & 7 \end{vmatrix}$$

Expand along 1st column

$$= 1 \cdot (10+7) + 1 \cdot (14-10)$$

$$= 17 + 4$$

$$= 21$$

2. (2 points) Is $A$ invertible? Briefly justify your answer.

Yes because $\det(A) \neq 0$.

3. (3 points) Find $\det(-A^T A^3)$. Show all your work.

$$\det(-A^T A^3) = (-1)^4 \det(A^T A^3)$$

$$= \det(A^T) \det(A)^3$$

$$= \det(A)^4$$

$$= 21^4$$

**Question 3.** (8 points) Let $A = \begin{bmatrix} 10 & -12 & 8 \\ 0 & 1 & 0 \\ -8 & 12 & -6 \end{bmatrix}$.

1. (3 points) Find the eigenvalues of A.

$$p(t) = \begin{vmatrix} 10-t & -12 & 8 \\ 0 & 1-t & 0 \\ -8 & 12 & -6-t \end{vmatrix} = (1-t)\begin{vmatrix} 10-t & 8 \\ -8 & -6-t \end{vmatrix}$$
$$= (1-t)(t^2 - 4t - 60 + 64)$$
$$= (1-t)(t^2 - 4t + 4) = (1-t)(t-2)^2$$

Eigenvalues $\leftrightarrow$ roots $\quad \lambda = 1$, $\lambda = 2$

2. (4 points) Find a basis for the eigenspace of each eigenvalue and state its dimension.

$\underline{\lambda = 1}$: $[A - I \,|\, 0] \sim \left[\begin{array}{ccc|c} 9 & -12 & 8 & 0 \\ 0 & 0 & 0 & 0 \\ -8 & 12 & -7 & 0 \end{array}\right]$

$R_1 = r_1 + r_3$ $\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -8 & 12 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 12 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_3$ free $\begin{bmatrix} -x_3 \\ -x_3/12 \\ x_3 \end{bmatrix} \to \left\{ \begin{bmatrix} -1 \\ -1/12 \\ 1 \end{bmatrix} \right\}$

$\underline{\lambda = 2}$: $[A - 2I \,|\, 0] \sim \begin{bmatrix} 8 & -12 & 8 \\ 0 & -1 & 0 \\ -8 & 12 & -8 \end{bmatrix} \sim \begin{bmatrix} 8 & -12 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & 1 & 0 \\ 0 & 1 & 0 & \\ 0 & 0 & 0 & \end{array}\right]$

$x_2 = 0$, $x_1 = -x_3$ $\to$ $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

3. (1 point) Is $A$ diagonalizable? Explain your answer.

No. $\dim(E_1) + \dim(E_2) = 1 + 1 = 2 < 3$

**Question 4.** (10 points) Let $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$. Diagonalize $A$ and use your answer to find $A^n$ for all integer $n$.

$p(t) = \det\begin{pmatrix} 1-t & 0 \\ 6 & -1-t \end{pmatrix} = (1-t)(-1-t) = t^2 - 1 = (t+1)(t-1)$

$\underline{\lambda = -1}$: $\left[\begin{array}{cc|c} 2 & 0 & 0 \\ 6 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \to x = 0$, $y$ free

eigenvector is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\underline{\lambda = 1}$: $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 6 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $3x = y$ $\to$ $\begin{bmatrix} x \\ 3x \end{bmatrix}$

eigenvector is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$P = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$, $P^{-1} = \frac{1}{-1}\begin{bmatrix} 3 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^n = (PDP^{-1}) \ldots (PDP^{-1}) = P D^n P^{-1}$

$= \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} (-1)^n & 0 \\ 0 & 1^n \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3\cdot(-1)^{n+1} & (-1)^n \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 3(-1)^{n+1} + 3 & (-1)^n \end{bmatrix}$

**Question 5.** (10 points) Fill in the blanks or circle Truth of False.

1. The rows of an $n \times n$ matrix of rank $n$ span $\mathbb{R}^n$. **T** or **F**.

2. If $A$ is an $m \times n$ matrix, then $\dim(\mathrm{Nul}(A)) + \mathrm{rank}(A) =$ <u>n</u>

3. A square matrix with a row consisting of zeroes is not invertible. **T** or **F**.

4. $\det \begin{bmatrix} \mathbf{r}_1 \\ 2\mathbf{r}_2 - \mathbf{r}_1 \end{bmatrix} = \underline{2} \det \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}$. Here $\mathbf{r}_1$ and $\mathbf{r}_2$ are row vectors in $\mathbb{R}^2$.

5. $\det(A + 2B) = \det(A) + 2\det(B)$ for all square matrices $A$ and $B$. **T** or **F**.

6. A matrix has a unique eigenvector for each eigenvalue. **T** or **F**.

7. $\det(P^{-1}AP) = \det(A)$. T

8. An $n \times n$ matrix with $n$ distinct eigenvalues is diagonalizable. **T** or **F**.

9. There exists an $n \times n$ matrix with $n+1$ distinct eigenvalues. **T** or **F**.

10. If $\mathbf{v}$ is an eigenvector of $A$ with eigenvalue $\lambda$, then it is an eigenvalue of $A^2$ with eigenvalue $\lambda^2$. **T** or **F**.