MATH 3300 - EXAM 2

October 17 2018

Name: Solutions

- No notes, calculators, other electronic devices are allowed.
- Full credit may not be given if sufficient justification is not provided.
- Academic Honesty Student Honor Code: "I will be academically honest in all of my academic work and will not tolerate academic dishonesty of others."
- This exam has 6 pages. You must check that no page is missing.

1	
2	
3	
4	
5	
Total	

Question 1. (8 points) Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 0 \\ 0 & 0 & -1 & 3 \end{bmatrix}$.

1. (3 points) Find the reduced echelon form (columns with leading 1s must have 0s everywhere else) of A. Show all your work.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 0 \\ 0 & 0 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. (2 points) Find a basis for the column space of A.

Leading 1s on columns 1, 3, 4

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \right\}$$

3. (2 points) Find a basis for the null space of A.

Free variable is x_2; top row: $x_1 = -2x_2$; middle row: $x_3 = 0$; bottom row: $x_4 = 0$

$$\sim \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} \rightsquigarrow \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

4. (1 point) What is rank(A))?

$\operatorname{rank}(A) = \dim(\operatorname{Col}(A)) = 3.$

Question 2. (8 points) Let $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 1 & 2 & 4 & 0 \\ 0 & 0 & -1 & 3 \\ 2 & 2 & 0 & 1 \end{bmatrix}$.

1. (4 points) Find det(A) using row or column operations. Show your work.

Column 1 = Column 2 − Column 1

$$\det(A) = \begin{vmatrix} 1 & 0 & -1 & 4 \\ 1 & 0 & 4 & 0 \\ 0 & 0 & -1 & 3 \\ 2 & 0 & 0 & 1 \end{vmatrix} = 0.$$

2. (2 points) Is A invertible? Briefly justify your answer.

No because det(A) = 0.

3. (2 points) Is dim(Col(A) = 4? Briefly justify your answer.

No. dim(Col(A)) = 4 is equivalent to det(A) ≠ 0.

Question 3. (8 points) Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

1. (3 points) Find the eigenvalues of A.

$p(t) = \det\begin{pmatrix} 1-t & 1 \\ 2 & 2-t \end{pmatrix} = (1-t)(2-t) - 2 = t^2 - 3t$
$= t(t-3)$

Eigenvalues $\leftrightarrow$ roots of $p(t)$

$\lambda = 0, 3$

2. (4 points) Find a basis for the eigenspace of each eigenvalue and state its dimension.

$\underline{\lambda = 0}$: $[A - 0I \mid 0] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

x_2 free, $x_1 = -x_2$ $\leadsto$ $\begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \sim \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ basis, dim = 1

$\underline{\lambda = 3}$, $[A - 3I \mid 0] \sim \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

x_1 free
$x_2 = 2x_1$ $\quad \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \leadsto$ basis $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$, dim = 1.

3. (1 point) Is A diagonalizable? Explain your answer.

Yes. Has 2 distinct eigenvalues.

Question 4. (8 points) Let $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$. Diagonalize A and use your answer to find A^n for all integer n.

$p(t) = \det(A - tI) = \det\begin{pmatrix} 2-t & -1 \\ -1 & 3-t \end{pmatrix} = (2-t)(3-t) - 0$

$= ~~t^2~~ ~~-5t~~$

$\underline{\lambda = 2}$ $\quad [A - 2I \mid 0] \sim \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \rightarrow$ x_1 free, $x_2 = 0$.

so eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\underline{\lambda = 3}$ $\quad \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad x_2$ free, $x_1 = -x_2$

so eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $\quad P^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$A^4 = (PDP^{-1}) \text{---} (PDP^{-1})$

$= P D^4 P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2^4 & 0 \\ 0 & 3^4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2^4 & 2^4 \\ 0 & 3^4 \end{bmatrix} = \begin{bmatrix} 2^4 & 2^4 - 3^4 \\ 0 & 3^4 \end{bmatrix}$

Question 5. (8 points) Fill in the blanks or circle Truth of False.

1. The null space of an $n \times n$ matrix of rank n is $\{\mathbf{0}\}$. **T** or **F**.

2. There is an 3×2 matrix with rank equal to 3. **T** or **F**.

3. If $\det(A) = 2$ and $\det(B) = 4$, then $\det(A^3(B^T)^{-1}) =$ 2

4. If I_n is the $n \times n$ identity matrix, then $\det(I_n) =$ 1

5. If $C = A + B$, then $\det(C) = \det(A) + \det(B)$. **T** or **F**.

6. For every diagonalizable A there is a unique diagonal matrix D and a unique invertible matrix P such that $A = PDP^{-1}$. **T** or **F**.

7. If $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then $3\mathbf{v}$ is also an eigenvector of A with eigenvalue λ

8. The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has (real) eigenvalues? **T** or **F**.