MATH 3300 - EXAM 1 PRACTICE

Name: Solutions

- No notes, calculators, other electronic devices are allowed.
- Full credit may not be given if sufficient justification is not provided.
- Academic Honesty Student Honor Code: "I will be academically honest in all of my academic work and will not tolerate academic dishonesty of others."
- This exam has 5 pages. You must check that no page is missing.

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Total	

Question 1. (10 points) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 1 \\ 0 & 4 & 1 \end{bmatrix}$. Read the Invertible Matrix Theorem.

1. (4 points) Find the reduced echelon form (columns with leading 1s must have 0s everywhere else) of A. Show all your work.

$$\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 1 \\ 0 & 4 & 1 \end{bmatrix} \overset{R_2 = r_2 - 3r_1}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \overset{R_2 = \frac{1}{2} r_2}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 4 & 1 \end{bmatrix}$$

$$\overset{R_3 = r_3 - 4r_2}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} \overset{R_3 = \frac{1}{5} r_3}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{R_1 = r_1 - r_3,\ R_2 = r_2 + r_3}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. (3 points) Are the columns of A linearly independent? Briefly justify your answer.

Yes, because, from ~~back substitution~~ row reduction, the only solution to $A\vec{x} = \vec{0}$ is a solution to $I_3 \vec{x} = \vec{0}$, which is $\vec{x} = \vec{0}$.

3. (3 points) Do the columns of A span $\mathbb{R}^3$? Briefly justify your answer.

Yes, the reduced echelon form of A has 3 leading 1s.

Question 2. (10 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y+z \\ x-z \end{bmatrix}.$$

Read the Invertible Matrix Theorem.

1. (2 points) Find the matrix A of T.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

2. (2 points) Find the reduced echelon form of A. Show all your work.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \overset{R_2 = r_2 - r_1}{\sim} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix} \overset{R_2 = -r_2}{\sim} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

3. (3 points) Is T one-to-one? Briefly justify your answer.

No. There is a non-trivial solution to $A\vec{x} = \vec{0}$. Looking at the reduced echelon form we set $x_3 = -1$, this gives $x_2 = 2$ and $x_1 = -1$.
Indeed $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

4. (3 points) Is T onto? Briefly justify your answer.

Yes, the row echelon form has 2 leading 1s.

Question 3. (10 points) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$. Show all your work.

1. (7 points) Find the inverse A^{-1} of A.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \overset{R_3 = r_3 - r_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\overset{R_2 = -r_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \overset{R_2 = r_2 + r_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\overset{R_1 = r_1 - r_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

← confirm this!

2. (3 points) Solve the linear system of equations $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Show all your work.

$$A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \vec{x} = A^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -1 \\ -1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

← confirm this!

Question 4. (10 points) Find the LU factorization of $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 3 \\ -1 & 2 & 1 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix} \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

Row reduce $A \overset{R_3 = r_3 + r_1}{\sim} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 3 \\ 0 & 5 & 1 \end{bmatrix}$. Used column $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. It starts with a 1 so $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is 1st column of L

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 3 \\ 0 & 5 & 1 \end{bmatrix} \overset{R_3 = r_3 - \frac{5}{2} r_2}{\sim} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & -\frac{13}{2} \end{bmatrix}$$

Used column $\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$. Turn it to $\begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$. Normalize it to $\begin{bmatrix} 0 \\ 1 \\ 5/2 \end{bmatrix}$ is the second column of L

echelon form. This is U.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 5/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & -\frac{13}{5} \end{bmatrix}$$

← confirm this!