MATH 3300 - EXAM 1

September 19 2018

Name: Solutions

- No notes, calculators, other electronic devices are allowed.
- Full credit may not be given if sufficient justification is not provided.
- Academic Honesty Student Honor Code: "I will be academically honest in all of my academic work and will not tolerate academic dishonesty of others."
- This exam has 5 pages. You must check that no page is missing.

1	
2	
3	
4	
Total	

Question 1. (10 points) Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 4 & -2 \\ 2 & -1 & -1 \end{bmatrix}$.

1. (4 points) Find the reduced echelon form (columns with leading 1s must have 0s everywhere else) of A. Show all your work.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 4 & -2 \\ 2 & -1 & -1 \end{bmatrix} \underset{R_2 = r_2 - 2r_1}{\overset{R_2 = r_2 - r_1}{\sim}} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & -1 \\ 0 & -3 & 1 \end{bmatrix}$$

$$\overset{R_3 = r_3 + r_2}{\sim} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_2 = \frac{r_2}{3}}{\sim} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{R_1 = r_1 - r_2}{\sim} \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

2. (3 points) Are the columns of A linearly independent? Briefly justify your answer.

No. The equation $A\vec{x} = \vec{0}$ has a non-trivial solution. For example, setting $x_3 = 3$ in the reduced echelon form we get $x_2 = \frac{x_3}{3} = 1$ and $x_1 = \frac{2}{3}x_3 = 2$. So $A\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \vec{0}$.

3. (3 points) Do the columns of A span $\mathbb{R}^3$? Briefly justify your answer.

No. The reduced echelon form has 2 leading 1s, which is fewer than 3.

Question 2. (10 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ 3z - y \end{bmatrix}.$$

1. (2 points) Find the matrix A of T.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix}$$

2. (2 points) Find the reduced echelon form of A. Show all your work.

$$A \overset{R_2 = -r_2}{\sim} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -3 \end{bmatrix} \overset{R_1 = r_1 - 2r_2}{\sim} \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -3 \end{bmatrix}$$

3. (3 points) Is T one-to-one? Briefly justify your answer.

No. The equation $A\vec{x} = \vec{0}$ has non-trivial solutions letting $x_3 = 1$ in the row echelon form we get
$x_2 = 3x_3 = 3$ $x_1 = -6x_3 = -6$.
So $A\begin{bmatrix} -6 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ⇒ T is not one-to-one.

4. (3 points) Is T onto? Briefly justify your answer.

Yes. The row echelon form of A has 2 leading 1s (= dimension of $\mathbb{R}^2$).

Question 3. (10 points) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$.

1. (6 points) Find the inverse A^{-1} of A.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \underset{R_3 = r_3 - r_1}{\overset{R_2 = r_2 - r_1}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\underset{R_3 = -r_3}{\overset{R_2 = -r_2}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$\underset{R_2 = r_2 - r_3}{\overset{R_1 = r_1 - r_3}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$\overset{R_1 = r_1 - 2r_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}}$$

2. (4 points) Let $B = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 0 & -2 \\ -1 & 3 & 2 \end{bmatrix}$. Find the 3×3 matrix X that satisfies the matrix equation $XA + B = 0$, where 0 is the 3×3 zero matrix. Show all your work.

$$XA = -B \longrightarrow X = -BA^{-1}$$

$$= -\begin{bmatrix} 3 & -2 & 1 \\ 0 & 0 & -2 \\ -1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= -\begin{bmatrix} 1 & 8 & -6 \\ -2 & 0 & 2 \\ 2 & -5 & 2 \end{bmatrix}$$

$$\boxed{X = \begin{bmatrix} -1 & -8 & 6 \\ 2 & 0 & -2 \\ -2 & 5 & -2 \end{bmatrix}}$$

Question 4. (10 points) Find the LU factorization of $A = \begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix}$. Show all you work.

Row reduce A

$$A = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix} \underset{R_3 = r_3 - 3r_1}{\overset{R_2 = r_2 + r_1}{\sim}} \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & -2 & 0 \end{bmatrix}$$

1st column of L is $\frac{1}{3}\begin{bmatrix} 3 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & -2 & 0 \end{bmatrix} \overset{R_3 = r_3 - \frac{2}{3} r_2}{\sim} \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -8 \end{bmatrix}$$

2nd column of L is $-\frac{1}{3}\begin{bmatrix} 0 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2/3 \end{bmatrix}$

echelon form so is U.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -8 \end{bmatrix}$$