MATH 3300 - Final

December 10 2018

Name: Answers

- No notes, calculators, other electronic devices are allowed.
- Full credit may not be given if sufficient justification is not provided.
- Academic Honesty Student Honor Code: "I will be academically honest in all of my academic work and will not tolerate academic dishonesty of others."
- This exam has 11 pages. You must check that no page is missing.

1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total	

Question 1. (10 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-y \\ z-y \\ x \end{bmatrix}.$$

1. (2 points) Find the matrix A of T.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

2. (2 points) Find the reduced echelon form of A. Show all your work.

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

3. (3 points) Is T one-to-one? Briefly justify your answer.

Yes. $Nul(A) = Nul(I) = \{0\}$

4. (3 points) Is T onto? Briefly justify your answer.

Yes. $Rank(A) = Rand(I) = 3$

Question 2. (10 points) Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix}$.

1. (6 points) Find the inverse A^{-1} of A.

$$A^{-1} = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

2. (4 points) Let $B = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}$. Find the 3×3 matrix X that satisfies the matrix equation $XA = B$. Show all your work.

$$X = BA^{-1} = \begin{bmatrix} 2 & -5 & 11 \\ 2 & -1 & 1 \\ 0 & -1 & 3 \end{bmatrix}$$

Question 3. (10 points) Find the LU factorization of $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 9 \\ 6 & 8 & 19 \end{bmatrix}$. Show all you work.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 4. (10 points) Let $A = \begin{bmatrix} 1 & 0 & 6 & 1 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 3 \\ 2 & 4 & 17 & 5 \end{bmatrix}$.

1. (3 points) Find the reduced echelon form (columns with leading 1s must have 0s everywhere else) of A. Show all your work.

$$A \sim \begin{bmatrix} 1 & 0 & 0 & -17 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. (2 points) Find a basis for the column space of A.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 1 \\ 17 \end{bmatrix} \right\}$$

3. (2 points) Find a basis for the null space of A.

$$\left\{ \begin{bmatrix} 17 \\ 3 \\ -3 \\ 1 \end{bmatrix} \right\}$$

4. (1 point) What is rank(A)?

$\text{rank}(A) = 3$

Question 5. (10 points) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 2 & 2 & 3 \end{bmatrix}$.

1. (4 points) Find $\det(A)$. Show your work.

$\det(A) = -4$

2. (2 points) Is A invertible? Briefly justify your answer.

Yes because $\det(A) \neq 0$

3. (2 points) Is $\dim(\text{Row}(A)) = 3$? Briefly justify your answer.

Yes because $\det(A) \neq 0$, so $\text{rank}(A) = 3$ and $\text{rank}(A) = \dim(\text{Row}(A))$.

Question 6. (10 points) Let $W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix} \right\}$. Find an orthogonal basis for W by using the Gram-Schmidt process.

$\vec{v}_3 = -\vec{v}_1 + \vec{v}_2$ so throw away $\vec{v}_3$.

Apply Gram-Schmidt to $\{\vec{v}_1, \vec{v}_2\}$ to get:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 0 \end{bmatrix} \right\}$$

Question 7. (10 points) Find the least-squares solution and the least squares error of the system

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

$$\vec{\hat{x}} = (A^T A)^{-1} A^T \vec{b}$$

$$= \frac{1}{14} \begin{bmatrix} 13 & -5 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \frac{1}{14} \begin{bmatrix} 11 \\ -1 \end{bmatrix}$$

$$\vec{\hat{x}} = \begin{bmatrix} \frac{11}{14} \\ \frac{-1}{14} \end{bmatrix}$$

Least squares error $= \left\| \begin{bmatrix} \frac{9}{14} \\ -\frac{3}{14} \\ \frac{6}{14} \end{bmatrix} \right\| = \frac{3}{14} \left\| \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \right\|$

$$= 3 \cdot \sqrt{14}$$

Question 8. (10 points) Diagonalize $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

$\lambda = -2, 2, 5$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \qquad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A = P D P^T$$

Question 9. (10 points) Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}$. Find the singular value decomposition of A.

$$A^T A = \begin{bmatrix} 6 & 4 \\ 9 & 5 \end{bmatrix}$$

Too hard...

Question 10. (10 points) Fill in the blanks.

1. If $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$, then

   $A^4 = \begin{bmatrix} 81 & 0 \\ 0 & 16 \end{bmatrix}$

2. If A and B are 2×2 matrices such that $\det(A) = -1$ and $\det(B) = 2$, then

   $\det(2A^3B^{-1}) = 2^2 (-1)^3 \, 2^{-1} = -2$

3. If $\mathbf{v}_1$ is an eigenvector of A with eigenvalue -2 and $\mathbf{v}_2$ is an eigenvector with eigenvalue 6, then

   $A(2\mathbf{v}_1 - \mathbf{v}_2) = 2(-2)\vec{v}_1 - 6\vec{v}_2 = -4\vec{v}_1 - 6\vec{v}_2$

4. If $\mathbf{u}$ is a vector in $\mathbb{R}^3$ and W is a subspace of $\mathbb{R}^3$ that is orthogonal to $\mathbf{u}$, then

   $\text{proj}_W(\mathbf{u}) = \vec{u}$

5. Let $W = \text{Span}\{\mathbf{e_1} + \mathbf{e_2}\}$ (the line $y = x$) in $\mathbb{R}^2$. Find its orthogonal complement.

   $W^\perp = \text{Span}\{\vec{e}_1 - \vec{e}_2\}$ the $y = -x$ line.