MATH 3300 - EXAM 3 PRACTICE

Name: Solutions

- No notes, calculators, other electronic devices are allowed.
- Full credit may not be given if sufficient justification is not provided.
- Academic Honesty Student Honor Code: "I will be academically honest in all of my academic work and will not tolerate academic dishonesty of others."
- This exam has 9 pages. You must check that no page is missing.

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2	
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5	
Total	

Question 1. (8 points) Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$. Find the matrix representing proj_W.

Basis is <u>not</u> orthogonal. Use Gram-Schmidt to obtain an orthogonal basis.

The second vector is $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \dfrac{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$$= \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \frac{-2}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 2/3 \\ 7/3 \end{bmatrix}$$

Use $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix} \right\}$ $\quad \left(\text{Length of } \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix} \right)^2 = 5^2 + 2^2 + 7^2 = 25 + 4 + 49 = 78 = 3 \cdot 26$

$$\text{proj}_W(\vec{e_1}) = \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ +1 \\ -1 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}}{\begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}} \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \frac{5}{78} \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}$$

$$= \frac{26}{78} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \frac{1}{78} \begin{bmatrix} 25 \\ 10 \\ 35 \end{bmatrix} = \frac{1}{78} \begin{bmatrix} 51 \\ 36 \\ 9 \end{bmatrix}.$$

Similarly $\text{proj}_W(\vec{e_2}) = \dfrac{26}{78} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \dfrac{1}{78} \begin{bmatrix} 10 \\ 4 \\ 14 \end{bmatrix} = \dfrac{1}{78} \begin{bmatrix} 36 \\ 30 \\ -12 \end{bmatrix}$

and $\text{proj}_W(\vec{e_3}) = -\dfrac{26}{78} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \dfrac{1}{78} \begin{bmatrix} 35 \\ 14 \\ 49 \end{bmatrix} = \dfrac{1}{78} \begin{bmatrix} 9 \\ -12 \\ 75 \end{bmatrix}$

So the matrix of proj_W is $\dfrac{1}{78} \begin{bmatrix} 51 & 36 & 9 \\ 36 & 30 & -12 \\ 9 & -12 & 75 \end{bmatrix}$

Question 2. (8 points) Let $W = \text{Span}\left\{ \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\3\\0\end{bmatrix}, \begin{bmatrix}3\\0\\0\\1\end{bmatrix} \right\}$. Find an orthogonal basis for W by using the Gram-Schmidt process.

$$\vec{v_1} = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}$$

$$\vec{v_2} = \begin{bmatrix}1\\0\\3\\0\end{bmatrix} - \frac{\begin{bmatrix}1\\0\\3\\0\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\\0\end{bmatrix}}{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\\0\end{bmatrix}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\3\\0\end{bmatrix} - \frac{4}{2}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$$

$$= \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$$

$$\vec{v_3} = \begin{bmatrix}3\\0\\0\\1\end{bmatrix} - \frac{\begin{bmatrix}3\\0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\\0\end{bmatrix}}{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\\0\end{bmatrix}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} - \frac{\begin{bmatrix}3\\0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}}{\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$$

$$= \begin{bmatrix}3\\0\\0\\1\end{bmatrix} - \frac{3}{2}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + \frac{3}{2}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

orthogonal basis $= \left\{ \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}$

Question 3. (8 points) Find the least-squares solution and the least squares error of the system

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

Solve $A^T A \vec{x} = A^T b$

$$A^T A = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$$

Now $\hat{\vec{x}} = (A^T A)^{-1} A^T \vec{b}$

$$(A^T A)^{-1} = \frac{1}{\det(A^T A)} \begin{bmatrix} 22 & 11 \\ 11 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & \frac{6}{11} \end{bmatrix}$$

$$\text{so } \hat{\vec{x}} = \begin{bmatrix} 2 & 1 \\ 1 & \frac{6}{11} \end{bmatrix} \begin{bmatrix} -4 \\ 11 \end{bmatrix} = \boxed{\begin{bmatrix} 3 \\ 2 \end{bmatrix} \leftarrow \text{least squares solution}}$$

$$A \hat{\vec{x}} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\left(\text{Least squares error}\right) = \|\vec{b} - A\hat{\vec{x}}\| = \left\| \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\| = \sqrt{3^2 + 1^2 + (-1)^2} = \sqrt{11}$$

Question 4. (8 points) Fit a linear function of the form $f(t) = \beta_0 + \beta_1 t$ to the data points $(0,1), (1,1), (2,2), (3,2)$, using least squares.

$$\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad \bar{x} = \frac{0+1+2+3}{4} = \frac{3}{2}$$

$$\vec{x}^* = \begin{bmatrix} 0-3/2 \\ 1-3/2 \\ 2-3/2 \\ 3-3/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

$$X^* = \begin{bmatrix} 1 & -3/2 \\ 1 & -1/2 \\ 1 & 1/2 \\ 1 & 3/2 \end{bmatrix}$$

First solve $\qquad X^* \vec{\beta}^* = \vec{y}$

$$\beta_0^* = \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \vec{y}}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} = \frac{6}{4} = \frac{3}{2}$$

$$\beta_1^* = \frac{\vec{x}^* \cdot \vec{y}}{\vec{x}^* \cdot \vec{x}^*} = \frac{2}{5}$$

So $\qquad y = \beta_0^* + \beta_1^* x^* = \frac{3}{2} + \frac{2}{5}\left(x - \frac{3}{2}\right)$

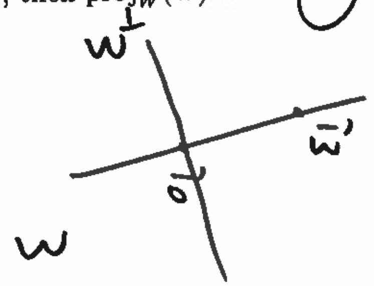
$$= \left(\frac{3}{2} - \frac{3}{5}\right) + \frac{2}{5}x = \frac{9}{10} + \frac{2}{5}x$$

$$\boxed{\beta_0 = \frac{9}{10}, \quad \beta_1 = \frac{2}{5}}$$

Question 5. (8 points) Fill in the blanks or circle Truth of False.

1. If $\mathbf{w}$ is in a subspace W, then $\text{proj}_W(\mathbf{w}) = \mathbf{w}$. **T** or **F**.

Picture:



no component of w along $W^\perp$

2. Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a set of vectors and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be the vectors obtained after Gram-Schmidt has been applied. The two sets of vectors have the same span. **T** or **F**.

3. Let $\mathbf{u}$ and $\mathbf{v}$ be two orthogonal vectors of length 3. Find the length of $\mathbf{u} + \mathbf{v}$.
$\|\mathbf{u} + \mathbf{v}\| =$ $\sqrt{2} \cdot 3$

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 = 3^2 + 3^2 = 18$$

$$\|\vec{u}+\vec{v}\| = \sqrt{18}$$

4. Let $W = \text{Span}\{\mathbf{e}_1\}$ be the 'x-axis is $\mathbb{R}^3$. What is the orthogonal complement of W?
$W^T =$ $\text{Span}\{\vec{e_2}, \vec{e_3}\}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $W^\perp$ if and only if $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$

$x = 0$

$\Rightarrow \begin{bmatrix} 0 \\ y \\ z \end{bmatrix}$ is the general element in $W^\perp$