Section 5.4: The Fundamental Theorem of Calculus.

We now get to the main theorem of Calculus I. The Fundamental Theorem of Calculus has two main versions.

Remember that from section 5.3, we define the *Indefinite Integral* of $f(x)$ on $[a,b]$ to be the area under the curve and we denote it as $\int_a^b f(x)dx$. We now introduce the theorems of this section:

Theorem 4 - The Fundamental Theorem of Calculus, Part 1:

If f is continuous on $[a,b]$, then $F(x)$ (defined below) is continuous on $[a,b]$ and differentiable on (a,b) and its derivative is $f(x)$. That is:

$$F(x) = \int_a^x f(t)dt \qquad \longrightarrow \qquad F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

A function defined in such a manner is whats called integral form. From section 5.3, this is defined as the area under the curve of $f(t)$ at the interval $[a,x]$. Moreover, this gives us that **all continuous functions** $f(x)$ **has an antiderivative.**

Theorem 4 (Continued) - The Fundamental Theorem of Calculus, Part 2:

If f is continuous on $[a,b]$, and if F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

This theorem allows us to evaluate the area under the curve for any arbitrary function; as long as the function is continuous on $[a,b]$, integration and evaluation will give us the answer. Ultimately, *this gives us a shortcut on finding the area under the curve without having to use rectangles to approximate.*

Note: Both of these theorems requires that the function $f(x)$ is **CONTINUOUS** on the interval $[a,b]$. If you force these theorems when it doesn't apply, you will get a bogus answer.

Example: Let $f(x) = 1/x$ on the interval $[-1,e]$. Sketch the graph of $f(x)$, then compute $\int_{-1}^e f(x)dx$.

With these precaution in mind, try the following:

Exercises: Use the Fundamental Theorem of Calculus, Part II (FTOC2) to compute the areas under the curve for the following functions and intervals:

1. $f(x) = \cos(x)$ at $[0,\pi/2]$
2. $g(x) = \sqrt{x}$ at $[1,25]$
3. $h(x) = x^2 + 5$ at $[0,3]$.

Exercises: Compute the following indefinite integrals:

1. $\int_0^{49} \left(x^2 + \sqrt{x}\right) dx$
2. $\int_1^{32} t^{-8/5} dt$
3. $\int_1^{\sqrt{12}} \frac{y^3 + \sqrt[3]{y}}{y^2} dy$

Computing the Derivative of a Functions in Integral Form:

Let $f(x) = \int_0^x (3t^2 + \cos(t))dt$. Compute $f'(x)$ two ways by using both parts of the FtoC.

Exercises: Compute the derivative of the following functions written in integral form.

1. $f(x) = \int_4^x t^7 \sin(9t)dt$
2. $g(x) = \int_x^3 5w^8 \cos(3w^2 + 4)dw$
3. $h(x) = \int_0^{3x^2} e^{8y+4} dy$
4. $j(x) = \int_{-2}^{e^{12x}} \sqrt[5]{9z^2 + 9} dz$
5. $p(x) = \int_x^{x^2} \arctan(41r + 4)dr$