5.2: Sigma Notation

Sigma Notation enables us to write a sum with many terms in the compact form:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_{n-1} + a_n.$$

This notation has three main parts. The first part is below the sigma. This includes your indexing variable k and your starting position, which is usually 0 or 1, but can be any integer.

The second part is above the sigma. This is your indexing variable's ending position, which is again any integer greater than your starting position. If n is $-\infty$, the sum is called an **infinite sum**.

The third part is to the right of the sigma: the formula for the kth term of the summands.

Compute the following:

1. $\displaystyle\sum_{k=1}^{4} k$ 2. $\displaystyle\sum_{k=0}^{5} 4k$ 3. $\displaystyle\sum_{k=1}^{3} 7$ 4. $\displaystyle\sum_{k=-2}^{3} (-1)^k k^2$

You should also be able to work backwards. Given a set of terms being summed, condense it into sigma notation.

Examples: Express the following sum in sigma notation. Note that there are many ways to do this correctly.

a. $\pi + 2\pi + 3\pi + 4\pi + 5\pi$ b. 10 + 8 + 6 + 4 + 2
c. 10 + 40 + 90 + 160 + 250 + 360 d. 1/4 + 1/5 + 1/6 + 1/7 + 1/8

Properties of Finite Sums:

Sum/Difference Rule: $\displaystyle\sum_{k=1}^{n}(a_k \pm b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

Constant Multiple Rule: $\displaystyle\sum_{k=1}^{n} c \cdot a_k = c\sum_{k=1}^{n} a_k$ Where c is any real number.

Constant Value Rule: $\displaystyle\sum_{k=1}^{n} c = nc$ Where c is any real value.

Suppose that $\displaystyle\sum_{k=1}^{n} a_k = 33$ and $\displaystyle\sum_{k=1}^{n} b_k = 45$.

Compute the following:

a. $\displaystyle\sum_{k=1}^{n} \frac{a_k}{15}$ b. $\displaystyle\sum_{k=1}^{n}(a_k + 4)$ c. $\displaystyle\sum_{k=1}^{n}(a_k - 3b_k)$

Sum of the first n integers, squares, and cubes: Section 5.2 presents four formulas in summation notation:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{(n^2)(n+1)^2}{4}$$

Examples: Use the formulas listed above to compute the following:

1. The sum of the first 37 positive integers.
2. The sum of the first 63 squares.
3. The sum of the first 19 positive cubes.

Riemann Sums Revisited: You may have noticed that as you use more and more rectangles, estimation with rectangles becomes a more and more precise way of computing the area under the curve.

Let's compute the Right Riemann Sum of $f(x) = 16 - x^2$ on the interval $[0, 4]$ using 100 rectangles of even width.

The width of each rectangle will be $\Delta x = \dfrac{4-0}{100} = 0.04$.

Remember that this width then partitions our interval [0, 4] into 101 partition points. Since we are computing the Right Riemann Sum, we will use all of these points except the first.

Thus, our area will be:

$$0.04 \cdot f(0.04) + 0.04 \cdot f(0.08) + 0.04 \cdot f(0.12) \quad + \quad \cdots \quad + \quad 0.04 \cdot f(3.92) + 0.04 \cdot f(3.96) + 0.04 \cdot f(4.00)$$

We will then use Sigma Notation (and its properties) to compute this summation, which is 100 rectangles added together.

Example 2: Compute the Right Riemann Sum of $f(x) = 4x^3$ on the interval $[0, 2]$ using 40 evenly spaced rectangles.