Section 2.5: Continuity.

A function $f(x)$ is said to be **continuous at a point** $x = a$ if **all three** of the following criteria are met:

- $f(a)$ is a real value.
- $\lim_{x\to a} f(x)$ is a real value.
- $\lim_{x\to a} f(x) = f(a)$.

When testing for continuity, you do not alter the function. Do not factor out things from the numerator/denominator. Do not multiply by the conjugate. Such tricks are reserved for limits.

Examples: Sketch the following piecewise functions and determine if each function is continuous at $x = 1$.

$$f(x) = \begin{cases} \sqrt{1-x} & \text{for } x < 1 \\ \ln(x) & \text{for } x > 1 \end{cases} \qquad g(x) = \begin{cases} x+5 & \text{for } x \leq 1 \\ 4 & \text{for } x > 1 \end{cases} \qquad h(x) = \begin{cases} 5 & \text{for } x = 1 \\ \dfrac{1}{x-1} & \text{for } x \neq 1 \end{cases}$$

Each example above illustrates the three types of **discontinuity.**

Type 1: Removable Discontinuity.
A function $y = f(x)$ is said to have a **removable discontinuity** at a point $x = a$ if **all three** of the following criteria are met:

- $\lim_{x\to a} f(x)$ exists (and is finite)
- $f(a)$ does not exist or $f(a) \neq \lim_{x\to a} f(x)$.

A **continuous extension of** $f(x)$ **at** $x = a$ can be created by creating a similar function $F(x)$ such that:

$$F(x) = \begin{cases} f(x) & \text{for } x \neq a \\ \lim_{x\to a} f(x) & \text{for } x = a \end{cases}$$

Examples: Create a continuous extension of $f(x)$ (defined above) that is continuous everywhere.

Type 2: Jump Discontinuity.
A function $y = f(x)$ is said to have a **jump discontinuity** at a point $x = a$ if **all three** of the following criteria are met:

- $\lim_{x\to a-} f(x)$ exists (and is finite)
- $\lim_{x\to a+} f(x)$ exists (and is finite)
- $\lim_{x\to a-} f(x) \neq \lim_{x\to a+} f(x)$

There is no way to "fix" this type of discontinuity.

One-Sided Continuity.

A function $f(x)$ is said to be **continuous from the left** at $x = a$ iif **all three** of the following criteria are met:

- $f(a)$ is a real value.
- $\lim_{x\to a^-} f(x)$ is a real value.
- $\lim_{x\to a^-} f(x) = f(a)$.

A function $f(x)$ is said to be **continuous from the right** at $x = a$ if **all three** of the following criteria are met:

- $f(a)$ is a real value.
- $\lim_{x\to a^+} f(x)$ is a real value.
- $\lim_{x\to a^+} f(x) = f(a)$.

Examples: Determine whether $g(x)$ is continuous from the left or continuous from the right at $x = 1$.

Type 3: Infinite Discontinuity.
A function $y = f(x)$ is said to have an **infinite discontinuity** at a point $x = a$ if **either** of the following criteria are met:

- $\lim_{x\to a-} f(x) = -\infty$ or $+\infty$

or

- $\lim_{x\to a+} f(x) = -\infty$ or $+\infty$

Examples: Using the graphs below, determine where each function is continuous. Then label the points of discontinuity as removable, jump, or infinite.