Section 2.2B: Limit Properties.

Review: Let $\dfrac{x^2+6x-7}{x-1}$. Compute $f(1)$ and $\lim\limits_{x\to 1} f(x)$.

Notice that even though $f(x)$ does not exist at a point, it is still possible for the limit to exist at that same point.

Now we start with the algebraic method of computing limits, starting with the following theorem:

Theorem 2 - Limits of Polynomials: If $P(x)$ is a polynomial, then $\lim\limits_{x\to c} P(x) = P(c)$.

Thus, to compute the limit of a polynomial at a point, all we have to do is evaluate it at one point; eliminating the need to tabulate form the left and from the right.

We also have the following properties of limits. Given real values L, M, c, and k, and if

$$\lim_{x\to c} f(x) = L \qquad \text{and} \qquad \lim_{x\to c} g(x) = M,$$

Sum Rule: $\lim\limits_{x\to c}(f+g)(x) = L + M$

Difference Rule: $\lim\limits_{x\to c}(f-g)(x) = L - M$

Product Rule: $\lim\limits_{x\to c}(f\cdot g)(x) = L\cdot M$

Constant Multiple Rule: $\lim\limits_{x\to c}(k\cdot f(x)) = k\cdot L$

Quotient Rule: $\lim\limits_{x\to c}(f/g)(x) = L/M$, with $M \neq 0$.

We can now delve into rational functions with these rules. Here's the first major theorem:

Theorem 3 - Limits of Rational Functions:
If $N(x)$ and $D(x)$ are polynomials, and $D(c) \neq 0$, then

$$\lim_{x\to c}\frac{N(x)}{D(x)} = \frac{N(c)}{D(c)}$$

Note that the main issue of concern is when a function evaluates to zero. If the numerator evaluates to zero, then it's not an issue. If the denominator evaluates to zero, we have a problem. The first case is:

Property 4 - Nonzero over Zero Property:
Given a rational function $R(x) = (N/D)(x)$, if $N(c) \neq 0$ and $D(c) = 0$, then $\lim\limits_{x\to c} R(x)$ does not exist.

To be able to use this property, it is imperative that the numerator does not evaluate to zero. If it does, that is if both $N(x)$ and $D(x)$ evaluates to zero, this is what's called **an indeterminate form**. This is technically an *undefined value* but it signals that we have more work to do to compute the value. An example of this situation is $h(x)$ at $x = 1$ from the previous page. Notice that the limit exists at these values, but the denominator evaluates to zero.

Dealing with indeterminate forms will involve the use of piecewise functions as well as two algebraic tricks: the factoring method and the conjugate trick.

Examples: Then determine the limit of the following functions for the value $x = 4$.

1. $f(x) = x^3 - 2x + 1$
2. $g(x) = \dfrac{9x}{3x-12}$
3. $h(x) = \dfrac{12-3x}{x^2-16}$
4. $j(x) = \dfrac{2-\sqrt{x}}{4-x}$

Section 2.4: More Limits.

Limits and Absolute Values.
Use the graph of $y = |x|$, as well as piecewise notation, to compute the following limits:

$$\lim_{x\to 4^-} \frac{|x-4|}{x^2-16} = \underline{\hspace{3cm}} \qquad \lim_{x\to 4^+} \frac{|x-4|}{x^2-16} = \underline{\hspace{3cm}} \qquad \lim_{x\to 4} \frac{|x-4|}{x^2-16} = \underline{\hspace{3cm}}$$

$$\lim_{x\to -3^-} \frac{x^2-9}{|x+3|} = \underline{\hspace{3cm}} \qquad \lim_{x\to -3^+} \frac{x^2-9}{|x+3|} = \underline{\hspace{3cm}} \qquad \lim_{x\to -3} \frac{x^2-9}{|x+3|} = \underline{\hspace{3cm}}$$

Trig Limits: Your knowledge with Pre-Calculus will be imperative in this section. You should familiarize yourself with this chart and the following trig properties.

x	0	$\pi/2$	π	$3\pi/2$	2π
$\sin(x)$					
$\cos(x)$					

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \cot(x) = \frac{\cos(x)}{\sin(x)} \qquad \cot(x) = \frac{1}{\tan(x)} \qquad \sec(x) = \frac{1}{\cos(x)} \qquad \csc(x) = \frac{1}{\sin(x)}$$

$$\sin^2(x) + \cos^2(x) = 1$$

We then look at the functions $f(x) = \sin(x)$ and $g(x) = \cos(x)$. Using the graphical approach, how do these functions behave as x goes to 0?

$$\lim_{x\to 0} \sin(x) = \qquad\qquad \lim_{x\to 0} \cos(x) =$$

Finally, we have the following theorem for trig limits:

Theorem 7: Given that θ is in radians, we have the following two limits:

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1 \qquad\qquad \lim_{\theta\to 0} \frac{\theta}{\sin(\theta)} = 1$$

Examples: Compute the following limits:

(a) $\displaystyle \lim_{h\to 0} \frac{h}{\sin(3h)} = \underline{\hspace{4cm}}$

(d) $\displaystyle \lim_{x\to 0} 6x^2(\cot(x))(\csc(2x)) = \underline{\hspace{4cm}}$

(b) $\displaystyle \lim_{t\to 0} \frac{\tan(15t)}{7t} = \underline{\hspace{4cm}}$

(e) $\displaystyle \lim_{w\to 0} \frac{w^2 - w + \sin(w)}{2w} = \underline{\hspace{4cm}}$

(c) $\displaystyle \lim_{y\to 0} 5y^2 \csc^2(13y) = \underline{\hspace{4cm}}$