NAME: ______________________________ GROUP WORK #8

For #1 and #2. Set up the following optimization problem: function in terms of one variable, then relevant domain.

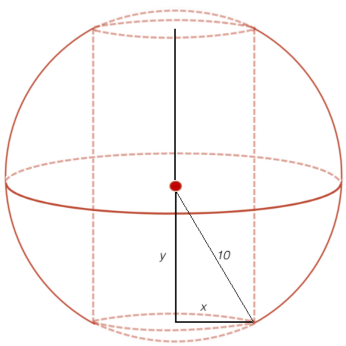
1. A fence is to be built to enclose a rectangular area of 240 sq ft. The fence along three sides is to be made of material that costs $5 per sq ft and the fourth side costs $16 per foot. Determine the dimensions of the enclosure that will minimize cost.

Let the rectangular fence have dimensions l = length and w = width. Let the fourth side be one of the widths. You are maximizing cost, where $C = 5l + 5l + 5w + 16w = 10l + 21w$.

Your constraint is $240 = l \cdot w$. Solving for l, you get $\frac{240}{w} = l$, and $C = 10\left(\frac{240}{w}\right) + 21w$.

Using your constraint equation, you'll get that the relevant domain is $(0, \infty)$. And your function $C(w)$ is continuous everywhere except when $w = 0$, which is not in your relevant domain.

2. Find the volume of the largest right circular **cylinder** that can be inscribed in a sphere of radius $r = 10$. (See if you can adjust our work from Monday.)



Let x = radius of the cylinder and y be the distance between the center of the sphere to the base of the cylinder. Using this diagram, you'll see that the height of the cylinder is $2y$.

The volume of a cylinder is $V = \pi r^2 h = \pi x^2(2y)$. And using the right triangle in the lower right, you'll get that $100 = x^2 + y^2$, so $100 - y^2 = x^2$.

$V(y) = \pi(100 - y^2)(2y)$. The relevant domain is $(0, 10)$, and your polynomial function is continuous everywhere, and thus is continuous on the relevant domain.

3. A right triangle whose hypotenuse is $x = \sqrt{6}$ metres long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.

Maximized Function: Volume $= \frac{1}{3}\pi r^2 h$. Constraint: $r^2 + h^2 = 6$

If you solve your constraint for r^2, you get $r^2 = 6 - h^2$. This gives you $V(h) = \frac{1}{3}\pi(6 - h^2)h = \frac{1}{3}\pi\left(6h - h^3\right)$.

Your relevant domain is $(0, \sqrt{6})$. You can set the constraint equal to zero to obtain it algebraically, or you can reason out that the hypotenuse must be the longest side.

The derivative is $V'(h) = \frac{1}{3}\pi\left(6 - 3h^2\right)$. This is never undefined, and is equal to zero when $h = \pm\sqrt{2}$. Only $\sqrt{2}$ is in the relevant domain.

Using the second derivative test, $V''(h) = \frac{1}{3}\pi\left(-6h\right)$. And since $V''(\sqrt{2})$ is negative, $V(h)$ is concave down around $h = \sqrt{2}$.

Thus, volume is maximized at $h = \sqrt{2}$, $r = 2$, and $V = \frac{4\sqrt{2}}{3}\pi$.

4. The U.S. Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 108 in. What dimensions will give a box with a square end the largest possible volume?

Let x be the width and the height of the box and l be the length of the box.

Maximized Function: Volume $= x^2 l$. Constraint: $4x + l = 108$

If you solve your constraint for l, you'll get $h = 108 - 4x$. This gives you $V(x) = x^2(108 - 4x) = -4x^3 + 108x^2$.

Your relevant domain is $(0, 27)$. You can se the constraint equal to zero to obtain it algebraically.

The derivative is $V'(h) = -12x^2 + 216x$, which is never undefined and is equal to zero when $x = 18$. This is our relevant critical point.

This breaks down your relevant domain into two subintervals: $(0, 18)$ and $(18, 27)$. Using representative points $x = 1$ and $x = 20$, you get $V'(1) = 204$ and $V'(20) = -480$. Since $V(x)$ goes from increasing to decreasing at $x = 18$, then that point is a local (and global) max of the volume function.

Thus, the volume is maximized when the width and height are 18 inches and the length is 36 inches.

5. A right circular cylinder has radius r and height h. The surface area of the cylinder, including top and bottom, is 460π sq. ft. Determine the dimensions that will maximize volume.

Maximizing Function: $V = \pi r^2 h$ **Constraint**: Surface Area $= 460\pi = 2\pi rh + 2\pi r^2$.

Solve the surface area for h to get $h = \dfrac{460\pi - 2\pi r^2}{2\pi r} = \dfrac{230 - r^2}{r}$. Then plug into the volume function to get $V(r) = \pi r^2 \left(\dfrac{230 - r^2}{r}\right)$. If you set the constraint equal to zero, you'll get a relevant domain of $\left(0, \sqrt{230}\right)$.

$V(r)$ can be simplified as $230\pi r - \pi r^3$. Thus, $V'(r) = 230\pi - 3\pi r^2$. This is never undefined, and is equal to zero when $h = \sqrt{\dfrac{230}{3}}$.

Using the second derivative test, $V''(r) = -6\pi$. And $V''\left(\sqrt{\dfrac{230}{3}}\right)$ will be negative. Thus, $V(r)$ is concave down at the critical point, which means that $V(r)$ has a maximum at $r = \sqrt{\dfrac{230}{3}}$.

The maximal volume is $230\pi\left(\sqrt{\dfrac{230}{3}}\right) - \pi\left(\sqrt{\dfrac{230}{3}}\right)^3$.