Section 2.4C: Building Functions

Answer the following questions. Again, be careful to use the appropriate independent variable. Then determine your relevant domain.

1. The sum of two numbers is 27. Let x denote one of the numbers. Express the sum of the square roots of the two numbers as a function of x.

   Let x and y be the two numbers. You are optimizing $S = \sqrt{x} + \sqrt{y}$. The constraint here is $x + y = 27$, so $x = 27 - y$. Thus, your optimized function is $S = \sqrt{27-y} + \sqrt{y}$. The relevant domain requires that both x and y be positive. If you set the constraint equal to zero, you'll get $(0, 27)$.

2. The point P lies on the graph of $y = 4x^3$. Given that point Q is on $(0, -3)$, determine the area and perimeter of the triangle shown as a function of x.

   As drawn in class, the base of the triangle is x, the height is $y + 3$ and the hypotenuse is $\sqrt{x^2 + (y+3)^2}$. Your optimized functions are listed below, and if P is in the first quadrant, the relevant domain is $(0, \infty)$.

   $A = \frac{1}{2}x(y+3) = \frac{1}{2}x\left(4x^3+3\right)$ and $P = \text{base} + \text{height} + \text{hypotenuse} = x + y + 3 + \sqrt{x^2 + (y+3)^2}$

   $P = x + 4x^3 + 3 + \sqrt{x^2 + (4x^3+3)^2}$.

3. A piece of wire of length 100cm is cut into two pieces and each piece is bent into a square. If x denotes the length of the first piece of wire, express the total area A enclosed by both squares.

   If x is the length of the first piece, then $100 - x$ is the length of the second piece. If each piece will be bent into a square, the first square will have sides $\frac{x}{4}$ cm long and the second square will have sides $\frac{100-x}{4}$ cm long. The total area is $A = \left(\frac{x}{4}\right)^2 + \left(\frac{100-x}{4}\right)^2$. The relevant domain is $(0, 100)$.

4. A rectangle has area 25 and length L. Express the perimeter P of the rectangle as a function of L.

   Perimeter $= 2L + 2W$. The constraint is $25 = WL$, so $\frac{25}{L} = W$. Thus the optimized function is $P = 2L + 2\left(\frac{25}{L}\right)$. The relevant domain is $(0, \infty)$.

5. The length of a rectangle is 5, and its width is x. If the rectangle is inscribed in a circle, find A, the area of the circle, as a function of x.

   The diagonal of the rectangle is equal to the diameter of the circle. Let d be the length of this side length. Then $d = \sqrt{5^2 + x^2} = \sqrt{25 + x^2}$. Thus the radius is $\frac{1}{2}\sqrt{25+x^2}$. In order for the rectangle to stay inside the circle, the relevant domain is $(0, 5)$.

6. A garden is in the shape of a rectangle. A sidewalk of width 6 surrounds the garden. One side of the garden has length 26 ft, while the other side has length x. Express the area of the sidewalk as a function of x.

   The area of the walkway can be computed by subtracting the big rectangle from the small rectangle. $A = (L+12)(W+12) - LW = 38(x+12) - 25x$. Since there is no constraint with x, the relevant domain is $(0, \infty)$.

7. A garden is in the shape of a circle of radius R. A sidewalk of width 6 surrounds the garden. Express the area of the sidewalk as a function of R.

   The area of the walkway is the larger circle minus smaller circle. The radius of the larger circle is $r + 6$, giving you $A = \pi(r+6)^2 - \pi r^2$. Again, there is no constraint on r, so the relevant domain is $(0, \infty)$.

8. The figure shows a semicircle of radius 3, with center at the origin and diameter along the x-axis, and an inscribed right triangle. Express the area of the triangle as as a function of x.

   As drawn in class, the base of the triangle is $x + 3$ and the height is y. Since the point (x, y) is on the circle in the first quadrant, then you have that $x^2 + y^2 = 9$, so $y = \sqrt{9 - x^2}$. Your optimized function is $A = \frac{1}{2}xy = \frac{1}{2}x\sqrt{9-x^2}$. The relevant domain is $(0, 3)$.

9. The figure shows a box with a square base. The volume of the box is 380 cubic feet. Express the surface area of the box, including top and bottom, as a function of x.

   If a box has a square base, then length = width. Thus, the volume is $V = x^2h$. The surface area is $2x^2 + 4xh$. Using the constraint $380 = x^2h$, $\frac{380}{x^2} = h$. Your optimized function is $A = 2x^2 + 4x\left(\frac{380}{x^2}\right)$. The relevant domain will be $(0, \infty)$.

10. A triangle ABC is inscribed in a square by joining one corner of the square with the midpoints of the two sides not containing the corner. If the edge length of the square is x, express the area of the triangle as a function on x.

    This question was skipped during class.

11. A silo is to be built as in the diagram, a cylinder surmounted by a hemisphere. The radius of the silo is $r = 70$ ft and the TOTAL silo height L is not yet determined. The cost to paint the outside of the entire silo is \$12 per square foot. Express the cost to paint the silo as a function of L.

    The height of the silo L = height of cylinder + height of hemisphere. Thus, $L = h + 70$. The cost function is $C = 12$(area of curved side) + 12(area of the hemispherical dome) $= 12(\pi r^2 h) + 12(0.5 \cdot 4\pi r^2)$. Plugging in $L - 70 = h$ and $r = 70$, you get the optimized function $C = 12(\pi(70)^2(L-70)) + 12(0.5 \cdot 4\pi(70)^2)$. The relevant domain is $(70, \infty)$: any values for L less than 70 gives you a negative height.

12. The figure shows a vitamin capsule constructed as a cylinder with a hemisphere attached to each end. The cylindrical part is $a = 1.3$ cm long and the radius r is not yet determined. Express the total volume of the capsule as a function of the radius r.

    The capsule is a cylinder with a hemisphere on each end. Thus the volume is a cylinder + a sphere. $V = \pi r^2 h + \frac{4}{3}\pi r^3$. The constraint is the length of the capsule: $a = 1.3 = r + h + r$, so $1.3 - 2r = h$. Your optimized function is $V = \pi r^2(1.3 - 2r) + \frac{4}{3}\pi r^3$. Your relevant domain is $(0, 0.65)$.

13. A right circular cylinder with radius r and height h. The surface area of the cylinder, including the top and bottom, is 460 sq ft. Express the volume of the cylinder as a function of r.

    Volume $= \pi r^2 h$. The constraint is $460 = 2\pi r^2 + 2\pi r h$. Solving for h gives you $h = \frac{460 - 2\pi r^2}{2\pi}$. Your optimized function is $V = \pi r^2 \left(\frac{460 - 2\pi r^2}{2\pi}\right)$. Your relevant domain is $\left(0, \sqrt{\frac{230}{\pi}}\right)$.

14. The large cone has radius 8 cm and height 54.4 cm. A frustum is created by removing the smaller cone of height y. Express the volume of the frustum as a function of y.

    If you draw in the radius and height of both the smaller and larger cone, you'll see similar triangles. This gives you a ratio of

    $$\frac{54.4}{8} = \frac{y}{r}$$

    . If you cross multiply, you'll get $54.4r = 8y$. The volume of the frustum is the area of the larger cone minus the smaller cone, so $V = \frac{1}{3}\pi(8)^2(54.4) - \frac{1}{3}\pi\left(\frac{8y}{54.4}\right)^2 y$. The constraint on y is $(0, 54.4)$.