Section 5.3: The Definite Integral.

We start this section with a review. Let $f(x)$ be a function defined on a closed interval $[a, b]$. Remember that:

$$\text{Left Riemann Sum } = \sum_{k=0}^{n-1} \triangle x \cdot f(a_k) \qquad \text{Right Riemann Sum} = \sum_{k=1}^{n} \triangle x \cdot f(a_k)$$

As more and more rectangles are used, the rectangular approximation becomes closer and closer to the actual area. If $\lim_{n\to\infty} \sum_{k=0}^{n-1} \triangle x \cdot f(a_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \triangle x \cdot f(a_k)$ and is finite, then $f(x)$ is said to be **Riemann integrable on** $[a, b]$**.**

This real valued limit, if it exists, is called the **definite integral of** $f(x)$ **on** $[a, b]$**.** and is denoted as:

$$\int_a^b f(x)dx$$

We then follow this up with a theorem:

If a function $f(x)$ is continuous on $[a, b]$, then its definite integral over $[a, b]$ exists.

Thus, every continuous function is Riemann Integrable.

Definite integrals have the following properties. Given that f and g are integrable over an interval $[a, b]$, we have:

1. Direction of Integration: $\int_b^a f(x)dx = -\int_a^b f(x)dx$
2. Zero Width Interval: $\int_a^a f(x)dx = 0$
3. Constant Multiple Rule: $\int_a^b k \cdot f(x)dx = k\int_a^b f(x)dx$
4. Sum and Difference Rule: $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
5. Additivity Rule with c between a and b: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Examples:

1. Given that f and g are integrable and that $\int_1^2 f(x)dx = 4$, $\int_2^5 f(x)dx = 12$, $\int_1^5 g(x)dx = 15.$

   Use the properties of integration to compute the following:

   (a.) $\int_7^7 f(x)dx$ (b.) $\int_1^5 f(x)dx$ (c.) $\int_5^1 g(x)dx$

   (d.) $\int_1^2 5f(x)dx$ (e.) $\int_1^5 (f(x) - g(x))dx$ (f.) $\int_1^2 (3f(x) + 2g(x))dx$

2. Given a function $f(x) = k$, where k is an arbitrary constant, determine the area under the curve of $f(x)$ on the interval $[a, b]$, with $0 \leq a \leq b$. That is, compute: $\int_a^b kdx$

3. Given the function $f(x) = x$, determine the area under the curve of $f(x)$ on the interval $[a, b]$, with $0 \leq a \leq b$. That is, compute: $\int_a^b xdx$