Sections 5.1: Area Under a Curve.
Given a function $y = f(x)$, the area under the curve of f over an interval $[a, b]$ is the area of the region bounded by the graph of the curve, the x-axis, and the vertical lines $x = a$ and $x = b$.

Sometimes this area is easy to calculate, as illustrated from the following examples. Compute the area under the curve of the following functions on the interval $[0, 2]$:

$f(x) = 5.$ $\quad$ $g(x) = |x|.$ $\quad$ $h(x) = 2x - 3.$ $\quad$ $j(x) = \sqrt{4 - x^2}.$

Sometimes, however, this isn't as easy as above. For the function $f(x) = x^2 + 4$ on the interval $[0, 2]$, the curved parabolic side inhibits us from using known geometric formulas to compute the area. We will later discuss four methods of estimating this area.

Partitions: We will first start by talking about the interval $[a, b]$. A **partition** of $[a, b]$ is a set of points $P = x_0, x_1, x_2, \ldots, x_n$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. Here's an example of a partition. The interval [0, 5] can be partitioned in many ways, one of which is by the set

$$P = \{0, 1, 2, 3, 4, 4.5, 4.75, 4.9999, 5\}.$$

A partition is then used to subdivide your interval. Thus, we can think of $[a, b]$ as:

$$[a, b] = [x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{n-1}, x_n]$$

And, in our earlier example, we have: $[0, 5] = [0, 1] \cup [1, 2] \cup [2, 3] \cup [3, 4] \cup [4, 4.5] \cup [4.5, 4.75] \cup [4.75, 4.9999] \cup [4.9999, 5]$.

Evenly spaced partitions: Although it isn't necessary, we would like to have these points evenly spaced out. In this situation, if you will have n subintervals of the interval $[a, b]$, then each subinterval will be of length (b-a)/n . This then gives a formula that specifies what each of the partition points will be.

If we let $\Delta x = \dfrac{b-a}{n}$, then the partition P_Δ will be: $P_\Delta = \{a, a + \Delta x, a + 2\Delta x, \ldots a + (n-1)\Delta x, b\}$.

Therefore, the partition points take the form $x_k = a + k\Delta x$, where x_k the k^{th} term of the partition. As a side note, you can think of a as $a + 0\Delta x$, and b as $a + n\Delta x$. This is part of why the subscripts were chosen earlier to denote a as x_0 and b as x_n.

Note: Most of the stuff we do in this class will use evenly spaced partitions.

Example: Consider the interval [0, 5]. Determine Δx for the interval [0, 5] for the values of n listed below, and create its corresponding partition, P_Δ.

(a.) $n = 2$ $\quad$ (b.) $n = 5$ $\quad$ (c.) $n = 10$

Estimating Area Under a Curve:
We have three methods of estimating the area under the curve: The Left Riemann Sum, The Right Riemann Sum, and The Midpoint Sum. All three methods will use rectangles.

We start with a function $y = f(x)$ and a closed interval $[a, b]$. Again, it is not mandatory that a partition of $[a, b]$ be evenly spaced, but for simplification purposes for this course, we will do so.

Example: Quadratic. Let $f(x) = x^2 + 3$. Consider the area under the curve of $f(x)$ on the interval $[0, 16]$. Answer the following:

(a) Partition your interval into two subintervals. Then use it to compute the **Left Riemann Sum**, **Right Riemann Sum**, and the **Midpoint Sum**.

(b) Partition your interval into four subintervals. Then use it to compute the **Left Riemann Sum**, **Right Riemann Sum**, and the **Midpoint Sum**.

Example: Rational. Let $f(x) = \dfrac{2520}{x}$. Consider the area under the curve of $f(x)$ on the interval $[1, 9]$. Answer the following:

(a) Partition your interval into two subintervals. Then use it to compute the **Left Riemann Sum**, **Right Riemann Sum**, and the **Midpoint Sum**.

(b) Partition your interval into four subintervals. Then use it to compute the **Left Riemann Sum**, **Right Riemann Sum**, and the **Midpoint Sum**.