Section 4.5: L'H$\hat{o}$pital's Rule.

Remember that an **Indeterminate Form** is attained when, while computing a limit, one substitutes the limiting value and obtain a certain value. One case that we have encountered is a "0/0" case. From Chapter 2, we know that a trick must be used to obtain this limit.

There are four indeterminate forms. They are:

- $\frac{0}{0}$
- $\frac{\infty}{\infty}$
- $0 \cdot \infty$
- $\infty - \infty$

Examples of each case:

$f(x) = \dfrac{x^2 + 2x}{x + 2}$ at $x = -2$

$g(x) = \dfrac{\tan(x)}{\sec(x)}$ at $x = \pi/2$

$h(x) = x^2 \cdot (-\ln(x))$ at $x = 0$

$j(x) = \dfrac{1}{x} - \dfrac{1}{x^2}$ at $x = 0$

Now that we have done the derivative rules, a new way of computing limits arise: L'H$\hat{o}$pital's Rule. Your knowledge of derivatives, as well as graphs of functions from Pre-Calculus will be vital in this section.

L'H$\hat{o}$pital's Rule: Suppose that $n(a) = d(a) = 0$ or both infinite. Suppose also that n and d are differentiable on an open interval I containing a and that $d'(x) \neq 0$ on I if $x = a$. Then:

$$\lim_{x \to a} \frac{n(x)}{d(x)} = \lim_{x \to a} \frac{n'(x)}{d'(x)}$$

provided that the limit on the right side of this equation exists.

There are a few things to remember when using L'H$\hat{o}$pital's Rule:

L'H$\hat{o}$pital's Rule vs Quotient Rule. L'H$\hat{o}$pital's Rule states that if your function $f(x)$ evaluates to an indeterminate form at $x = a$, taking the *separate derivatives of the numerators and the denominators*, then evaluating your function will result in your limit. Using L'H$\hat{o}$pital's Rule properly should result in $\lim\limits_{x \to a} f(x)$.

Make sure that the theorem applies. If your function does not evaluate to an indeterminate form, using L'H$\hat{o}$pital's Rule will give you an incorrect answer. Consequently, it can be used iteratively as long as the theorem applies.

(1) $\displaystyle \lim_{x \to 0} \frac{5}{\cos(x)}$

(2) $\displaystyle \lim_{x \to 0} \frac{\sin(8x) + x}{e^x}$

Nonzero over Zero property still applies. This is a followup for the previous note. If (before or after) using L'H$\hat{o}$pital's Rule, you get a nonzero over zero case, then the limit does not exist.

(3) $\displaystyle \lim_{x \to 0} \frac{4x^2 - 1}{\sin(x)}$

(4) $\displaystyle \lim_{x \to 0} \frac{4x}{\cos(x) - 1}$

L'H$\hat{o}$pital's Rule requires a numerator and a denominator. In order to use L'H$\hat{o}$pital's Rule, you must have it in the right form.

(5) $\displaystyle \lim_{x \to \infty} xe^{-x}$

(6) $\displaystyle \lim_{x \to 0} -x^2 \ln(x)$

L'H$\hat{o}$pital's Rule isn't always efficient. Sometimes it is better to not use L'H$\hat{o}$pital's Rule.

(7) $\displaystyle \lim_{x \to \infty} \frac{4x^2 + 3x + 1}{5x^2 - 17}$

(8) $\displaystyle \lim_{x \to 0} \tan(8x) \csc(7x)$

There's also the concept of an **Indeterminate Powers**, where you have exponential functions involved. They are:

- 0^0
- 1^∞
- ∞^0

L'Hôpital's Rule on Indeterminate Powers: If we have that

$$\lim_{x\to a} \ln(f(x)) = L, \qquad \text{then} \qquad \lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln(f(x))} = e^L.$$

Here, a may be either infinite or finite.

There are a few things to remember when using L'Hôpital's Rule and Indeterminate Powers:

1. Make sure that you actually have an Indeterminate Power. Make sure you know the difference between the coefficient and the base. Remember that the constant multiple rules apply for limits.

   (9) $\lim_{x\to 0}(x+1)^{3\sin(x)}$.   (10) $\lim_{x\to 0^+}(\tan(x))^{3x}$

2. Using your function $f(x)$, take the natural log of both sides. Then check to see if the new function satisfies L'Hôpital's Rule. Don't forget that at the end, if this limits to L, then the original limit will be e^L.

   (11) $\lim_{x\to 0}(4x+1)^{5/x}$.   (12) $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n$