Section 4.4: Concavity and Curve Sketching

Definition: The graph of a differentiable function $y = f(x)$ is:

1. **Concave Up** at $x = c$ if the graph of $f(x)$ lies *above* the tangent line at $x = c$.
2. **Concave Down** at $x = c$ if the graph of $f(x)$ lies *below* the tangent line at $x = c$.

For a function $f(x)$, an **inflection point** is a point on the graph of $f(x)$ where concavity changes.

Concavity and Second Derivatives:
Much like the first derivative tells us where your function increases/decreases, the second derivative will tell us where your function will be concave up/concave down.

For a differentiable function $f(x)$, its inflection point(s) will occur where your second derivative is zero or undefined. These possible critical points will again break up your domain into subintervals.

Example: Determine for which intervals $f(x) = \dfrac{1}{20}x^5 - \dfrac{1}{6}x^4 + \dfrac{2}{3}x + \dfrac{4}{7}$ is concave up/down. Then determine the inflection points of $f(x)$.

Determining Absolute Extrema:
There are three things to consider when dealing with absolute extrema. They are as follows:

- Does your function have any vertical asymptotes?
- How does your function behave at $-\infty$ and ∞? L'Hopital's Rule might help here.
- If your graph does not go to $\pm\infty$, then your lowest local minimum is your global minimum and your highest local maximum is your highest global maximum.

Curve Sketching: Tying sections 4.3 − 4.5 together, we can then get a pretty accurate sketch of many functions.

Strategy for Curve Sketching of $y = f(x)$:

1. Identify the domain of f. Then determine if it has any asymptotes.
2. Compute $f'(x)$ and $f''(x)$. Simplify as you go to make your work easier.
3. Find the possible critical points of f by analyzing $f'(x)$.
4. Determine increasing/decreasing intervals and label CPs.
5. Find the possible inflection points of f by analyzing $f''(x)$.
6. Determine concave up/down intervals.
7. Plot key points, such as critical points and inflection points and sketch. Remember that when you're sketching, CPs and IPs are plugged into *the original function*, since you're plotting points of $f(x)$.

Exercises: Give a rough sketch of the four functions listed below:

(a) $f(x) = x^4 - 12x^3$

(b) $g(x) = \dfrac{5x}{x+1} - 4$

(c) $h(x) = xe^{-x}$

(d) $j(x) = x^{2/3}\,(22.5 - 3x)$

Example 3: Give a rough sketch of the function $h(x) = xe^{-x}$. Determine for which intervals $h(x)$ is increasing/decreasing, concave up/down. Find all IPs and CPs and label all CPs as local max, local min, or not an extrema. Finally, determine if $h(x)$ has any absolute extrema.

Domain: First off, the domain of $h(x)$ is $(-\infty, \infty)$. Since $h(x)$ is a product of a polynomial (x) and an exponential function e^{-x}, both of which are continuous everywhere, $h(x)$ is also continuous everywhere.

First Derivative Part: Using the product rule, $h'(x) = e^{-x} - xe^{-x}$. If you factor out e^{-x}, you get $h'(x) = e^{-x}(1-x)$.

The first derivative is never undefined. The derivative is equal to zero when each factor equal to zero. Thus, $x = 1$ is a possible Critical Point.

So now we use the first derivative analysis. Picking representative points $x = 0$ and $x = 2$, we have:

$f'(0) = e^{-0}(1-0) = 1$. (positive) $\qquad$ $f'(2) = e^{-2}(1-2) = -1/e^2$. (negative)

Thus, $h(x)$ is increasing at $(-\infty, 1)$ and decreasing from $(1, \infty)$ and $x = 1$ is a **local max**.

Second Derivative Part: Using the addition and product rules, we have $h''(x) = -e^{-x} - e^{-x} + xe^{-x}$. Again, you can simplify this to $h''(x) = e^{-x}(-2 + x)$.

The second derivative is again never undefined. The second derivative is equal to zero when $x = 2$, which is our possible Inflection Point.

Now we do the second derivative analysis. Picking representative points $x = 0$ and $x = 3$, we have:

$f''(0) = e^{-0}(-2+0) = -2$. (negative) $\qquad$ $f''(3) = e^{-3}(-2+3) = 1/e^3$. (positive)

Thus, $h(x)$ is concave down at $(-\infty, 2)$ and concave up at $(2, \infty)$ and $x = 2$ is an inflection point.

Tying it Altogether: INC, CD: $(-\infty, 1)$. DEC, CD: (1, 2). DEC, CU: $(2, \infty)$.

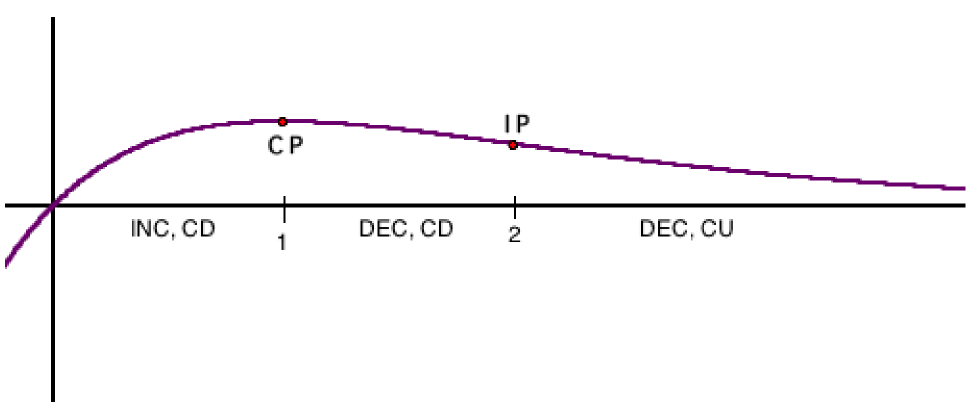
Behavior at the Tail Ends: You have to analyze what your function does at $\pm\infty$:

$$\lim_{x\to-\infty} \frac{x}{e^x} \longrightarrow \frac{-\infty}{+0} = \text{undefined (nonzero/zero.)} = -\infty$$

$$\lim_{x\to\infty} \frac{x}{e^x} \longrightarrow \frac{\infty}{\infty} = \text{indeterminate (use L'Hopital.)} = \lim_{x\to\infty} \frac{1}{e^x} \longrightarrow \frac{1}{\infty} = 0.$$

Thus, we will have an absolute max. This is because of two things: $h(x)$ is continuous and because $h(x)$ does not have a tail end going to $+\infty$. Note that if your function has vertical asymptotes, it does not matter what the tail ends do since it already goes to infinity in the middle.

Sketching the Function:



Example 4: Give a rough sketch of the function $j(x) = x^{2/3}(22.5 - 3x)$. Determine for which intervals $j(x)$ is increasing/decreasing, concave up/down. Find all IPs and CPs and label all CPs as local max, local min, or not an extrema. Finally, determine if $j(x)$ has any absolute extrema.

Domain: Again, the domain is $(-\infty, \infty)$. Your function is a product of $x^{2/3}$ (continuous by the Odd Root Law), and $(22.5 - 3x)$, which is a polynomial. Thus $j(x)$ is continuous everywhere.

First Derivative Part: First off, you should simplify your function to $j(x) = 22.5x^{2/3} - 3x^{5/3}$. This will save you a lot of work since now it's just a power rule. Thus:

$$j'(x) = \frac{45}{3}x^{-1/3} - \frac{15}{3}x^{2/3} = 15x^{-1/3} - 5x^{2/3}$$

If you factor the common multiple (5) and the lowest power of x (which is $-1/3$), we get:

$$j'(x) \quad = \quad 5x^{-1/3}(3 - x) \quad = \quad \frac{5(3-x)}{\sqrt[3]{x}}$$

Notice that this is undefined when $x = 0$ since your first term is really a $\sqrt[3]{x}$ in the denominator. This gives us our first possible Critical Point. Our other critical point is at $x = 3$.

Using our representative points $x = -1$, $x = 1$, and $x = 8$, we have:

$$j'(-1) = \frac{5(3+1)}{\sqrt[3]{-1}} = \frac{20}{-1} = \text{negative}$$

$$j'(1) = \frac{5(3-1)}{\sqrt[3]{1}} = \frac{10}{1} = \text{positive}$$

$$j'(5) = \frac{5(3-5)}{\sqrt[3]{8}} = \frac{-10}{2} = \text{negative}$$

Thus, $f(x)$ is increasing at $(0, 4)$ and decreasing at $(-\infty, 0) \cup (4, \infty)$. A local min occurs at $x = 0$ and a local max occurs at $x = 4$.

Second Derivative Part: Using our unsimplified first derivative, we get that, after simplification:

$$j''(x) \quad = \quad -\frac{15}{3}x^{-4/3} - \frac{10}{3}x^{-1/3} \quad = \quad -\frac{5}{3}x^{-4/3}(3 + 2x)$$

We again have a possible undefined value at $x = 0$ and our other possible inflection point is at $x = -1.5$.

Using our representative points $x = -8$, $x = -1$, and $x = 1$, we have:

$$j''(-8) = \frac{-5(3-16)}{3\sqrt[3]{(-8)^4}} = \frac{-65}{36} = \text{negative}$$

$$j''(-1) = \frac{-5(3-4)}{3\sqrt[3]{(-1)^4}} = \frac{5}{3} = \text{positive}$$

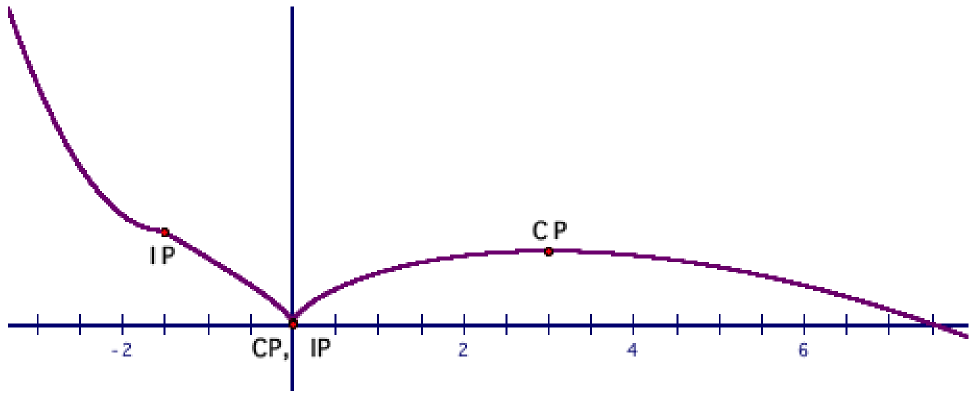
$$j''(1) = \frac{-5(3+2)}{3\sqrt[3]{(1)^4}} = \frac{-25}{3} = \text{negative}$$

Thus, $x = -2$ and is an inflection point, whereas $x = 0$ is not. $f(x)$ is concave up at $(-\infty, -2)$ and concave down from $(-2, \infty)$.

Tying it together: DEC, CU: $(-\infty, -2)$. DEC, CD: $(-2, 0)$. INC, CD: (0, 4), DEC, CD: $(4, \infty)$.

Behavior at Tail Ends: This function has range $(-\infty, \infty)$, thus we will not have absolute extrema.

Sketching the Function: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