Section 4.2: The Mean Value Theorem.

We start this section by recapping two key terms: **continuity** and **differentiability**.

A function $f(x)$ is said to be **continuous at a point** $x = c$ if the following three things are satisfied:

- $f(c)$ exists
- $\lim_{x \to c} f(x)$ exists
- these two values are equal.

A function $f(x)$ is said to be **differentiable at a point** $x = c$ if $f'(c) = \lim_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$ exists.

Property #1: Differentiability Implies Continuity.

If $f(x)$ is differentiable at $x = c$, then it must also continuous at $x = c$.

Proof: Since $f(x)$ is differentiable at $x = c$, we can assume that $f'(c)$ exists and that :

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

We can also assume that $f(c)$ exists. Otherwise the numerator on the right side would be undefined.
Let $x = c + h$. Then $h = x - c$. This gives us:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \frac{\lim_{x \to c} f(x) - f(c)}{\lim_{x \to c} x - c}$$

Multiply both sides by the denominator and you get:

$$\left(\lim_{x \to c} x - c\right) \cdot f'(c) = \lim_{x \to c} f(x) - f(c)$$

And as x approaches c, the left limit goes to zero. This then gives us $\lim_{x \to c} f(x) = f(c)$. Thus, we have continuity.

Example: Let $f(x) = \sqrt[3]{x}$. Determine where $f(x)$ is continuous and differentiable. Does this contradict Property #1?

The Mean Value Theorem:
If $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists at least one value $x = c$ such that:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Graphically, this means that on a given interval, the average rate of change of $f(x)$ on that interval will equal to the instantaneous rate of change at some point inside that interval.

We will revisit this theorem after Test 3. But for now it is a vital part of the next section.

Increasing and Decreasing Intervals.
Suppose that the interval I is a subset of the domain of a function $f(x)$.
The function $f(x)$ is considered to be **increasing** on the interval I, if for every $x_1 < x_2$ in I, $f(x_1) < f(x_2)$.
The function $f(x)$ is considered to be **decreasing** on the interval I, if for every $x_1 < x_2$ in I, $f(x_1) > f(x_2)$.

4.3: Monotonic Functions and the First Derivative Test

Corollary 3: The Derivative and Increasing/Decreasing.
Suppose that $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b). If $f'(x)$ is positive for all x in (a,b), then $f(x)$ is increasing on (a,b). If $f'(x)$ is negative for all x in (a,b), then $f(x)$ is decreasing on (a,b).

Proof: Because $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then the Mean Value Theorem applies. Let $x_1 < x_2$ be any two points in $[a,b]$. the MVT then states that for some c between x_1 and x_2,

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) \qquad \longrightarrow \qquad f(x_2)-f(x_1) = f'(c)\cdot(x_2-x_1)$$

Therefore, $f(x_2) = f(x_1) + f'(c)\cdot(x_2-x_1)$. Since $x_1 < x_2$, we know that $x_2 - x_1$ is positive.

If $f'(c)$ is positive, then $f(x_2)$ is bigger than $f(x_1)$. This tells us that $f(x)$ is increasing on that interval.

If $f'(c)$ is negative, then $f(x_2)$ is smaller than $f(x_1)$. This tells us that $f(x)$ is decreasing on that interval.

Examples: You are given the *derivatives* of three functions below. Determine the intervals for which the original functions are increasing and decreasing.

1. $f'(x) = 3x^2 + 9x + 12$
2. $g'(x) = 2e^x + 2xe^x$
3. $h'(x) = \dfrac{6+2x}{\sqrt[3]{x-1}}$

Examples: Determine the intervals for which the following functions are increasing/decreasing.

1. $f(x) = x^4 - 108x^3$
2. $g(x) = \dfrac{\ln(2x)}{x}$