MATH 8030 Homological Algebra — **Problem Set 3**, due May 6 by end of the day

1. Let I denote the category

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$$

   and let $\mathcal{A}$ be an abelian category. Prove that the colimit of any functor $F : I \to \mathcal{A}$ exists. (This colimit is called a *pushout*.)

2. (Do not hand in) Weibel, Exercise 3.2.1

3. Weibel, Exercise 3.2.2, 3.4.1

4. Let A and B denote cochain complexes of objects in an abelian category $\mathcal{A}$. Let $\mathcal{H}om(A, B)$ denote the total Hom cochain complex. Prove $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A, B[n]) = \mathrm{H}^n(\mathcal{H}om(A, B))$.

5. Consider the category $\mathbf{Ab}$ of abelian groups. Compute $\mathrm{Hom}_{\mathbf{T}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}[n])$ for all n for $\mathbf{T} = \mathbf{K}(\mathbf{Ab})$ and $\mathbf{T} = \mathbf{D}(\mathbf{Ab})$.

6. (Not to hand in) Let $R = \mathbb{C}[\mathbf{t}, \mathbf{t}^{-1}]$, the ring of Laurent polynomials in one variable. An R-module is the same as a complex vector space V equipped with an automorphism $\mathbf{t} : V \to V$. Given an R-module M, let $M^{\mathbf{t}}$ be the space of $\mathbf{t}$-invariants in M:

$$M^{\mathbf{t}} = \{m \in M | \mathbf{t}m = m\}.$$

   Given two R-modules M,N, consider the space $\mathrm{Hom}_{\mathbb{C}}(M, N)$ of linear transformations between them. This can be made into an R-module as follows: for $f \in \mathrm{Hom}_{\mathbb{C}}(M, N)$, let $(\mathbf{t} \cdot f)(m) = \mathbf{t}f(\mathbf{t}^{-1}m)$. Show that there is a natural isomorphism

$$\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_{\mathbb{C}}(M, N)^{\mathbf{t}}.$$

7. Now prove the derived version of the previous result.

   (a) Explain how to define $R\mathrm{Hom}_{\mathbb{C}} : \mathbf{D}^-(R - \mathrm{mod})^{op} \times \mathbf{D}^+(R - \mathrm{mod}) \to \mathbf{D}^+(R - \mathrm{mod})$.

   (b) Let $J : R - \mathrm{mod} \to \mathbb{C} - \mathrm{mod}$ be the functor $J(M) = M^{\mathbf{t}}$. Show that J is left exact.

   (c) Prove that for $M \in \mathbf{D}^-(R - \mathrm{mod})$ and $N \in \mathbf{D}^+(R - \mathrm{mod})$, there is a natural isomorphism

$$R\mathrm{Hom}_R(M, N) \cong RJ(R\mathrm{Hom}_{\mathbb{C}}(M, N)).$$

   Hint: You will need to show that if $A, I \in R - \mathrm{mod}$ with I injective, then $\mathrm{Hom}_{\mathbb{C}}(A, I)$ is also an injective R-module.