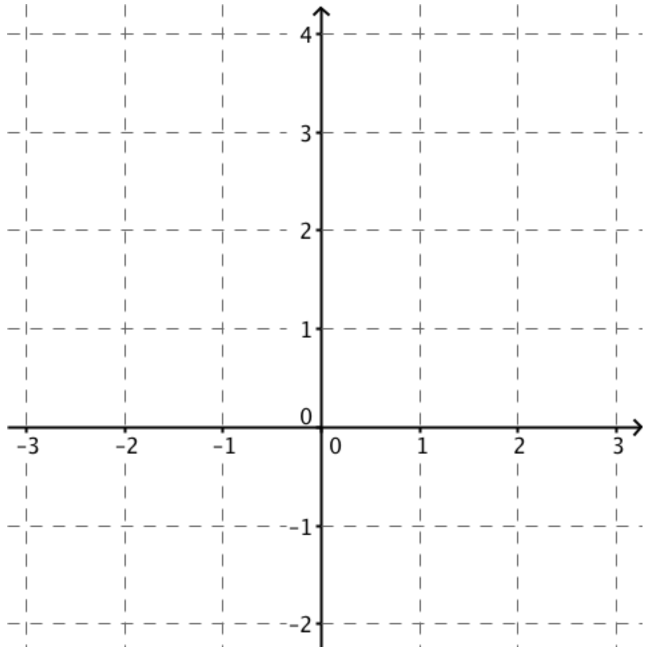
1. Sketch the graph of $f(x) = \begin{cases} x+2 & x \neq -1 \\ 3 & x = -1 \end{cases}$ on the axes below. What happens to the y-values on the graph of $y = f(x)$ as the x-values get closer and closer to 0 (but never get there)?



$$\lim_{x \to -1} f(x) =$$

"The limit as x approaches -1, of $f(x)$ is ____________."

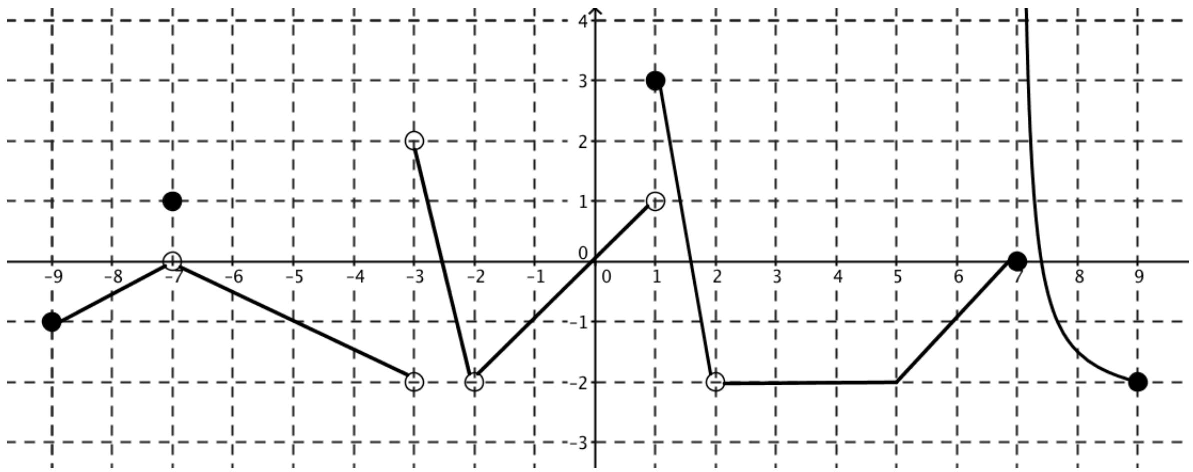
Definition. We write $\lim_{x \to c} f(x) = L$ and say, "The limit as x approaches c, of $f(x)$, equals L," if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to c, on either side of c, but *not equal* to c.

2. Complete the table below; make sure your calculator is in *radian mode*.

x	0.01	0.001	0	-0.001	-0.01
$\frac{\sin(x)}{x}$					

Using the information in the table, make an educated guess for $\lim_{x \to 0} \frac{\sin(x)}{x}$:
(We can't be sure – yet – that our guess is correct.)

3. Below is the graph of $y = f(x)$. Use it to determine the following limits.



$\lim_{x \to -7} f(x)$ $\qquad$ $\lim_{x \to -3} f(x)$ $\qquad$ $\lim_{x \to 1} f(x)$ $\qquad$ $\lim_{x \to 5} f(x)$

<u>On your paper.</u> Make rough sketches of the graphs of $f(x) = x$, $f(x) = 2$ and $f(x) = x + 2$.

<u>Limit Laws</u> If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ exist, then:

	<u>Example</u>
Identity Rule: $\lim_{x \to c} x = c$	
Constant Rule : $\lim_{x \to c} k = k$ (k is a constant)	
Sum Rule: $\lim_{x \to c} (f(x) + g(x)) = L + M$	
Difference Rule: $\lim_{x \to c} (f(x) - g(x)) = L - M$	
Constant Multiple Rule: $\lim_{x \to c} k(f(x)) = k \cdot L$ (k is a constant)	
Product Rule: $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$	
Quotient Rule: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ *provided that $M \neq 0$	
Power Rule: $\lim_{x \to c} [f(x)]^n = L^n$	
Root Rule: $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$ *provided n is a positive integer, and $L > 0$ if n is even	

Limits of Polynomials and Rational Functions: If $P(x)$ and $Q(x)$ are *polynomials* and $\lim_{x\to c} Q(x) \neq 0$, then $\lim_{x\to c} P(x) = P(c)$ and $\lim_{x\to c} \dfrac{P(x)}{Q(x)} = \dfrac{P(c)}{Q(c)}$.

4. $\lim_{x\to 2}(x^3 - 17x + 23)$

5. $\lim_{x\to -1} \dfrac{(x-5)(2x+1)}{(x^2-3)^3}$

Limits of Trig Functions: If c is a number *in the domain of* a trig function f, then $\lim_{x\to c} f(x) = f(c)$.

6. $\lim_{\theta\to \pi/2} \sin^2(\theta)$

Additional Examples

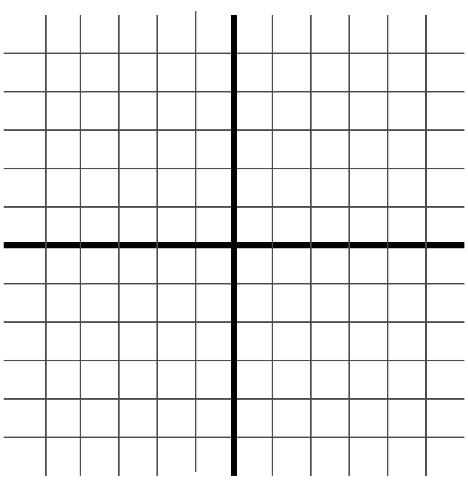
7. If $\lim_{x\to 3} f(x) = 5$ and $\lim_{x\to 3} g(x) = -4$, determine $\lim_{x\to 3}(2g(x) - 7f(x))$.

8. $\lim_{x\to 3} \dfrac{x^3 - x^2 - 6x}{x^2 - 4x + 3}$

9. $\lim_{h\to 0} \dfrac{\sqrt{2h+13} - \sqrt{13}}{h}$

10. $\lim_{y\to 49} \dfrac{y - 49}{\sqrt{y} - 7}$

The Sandwich Theorem

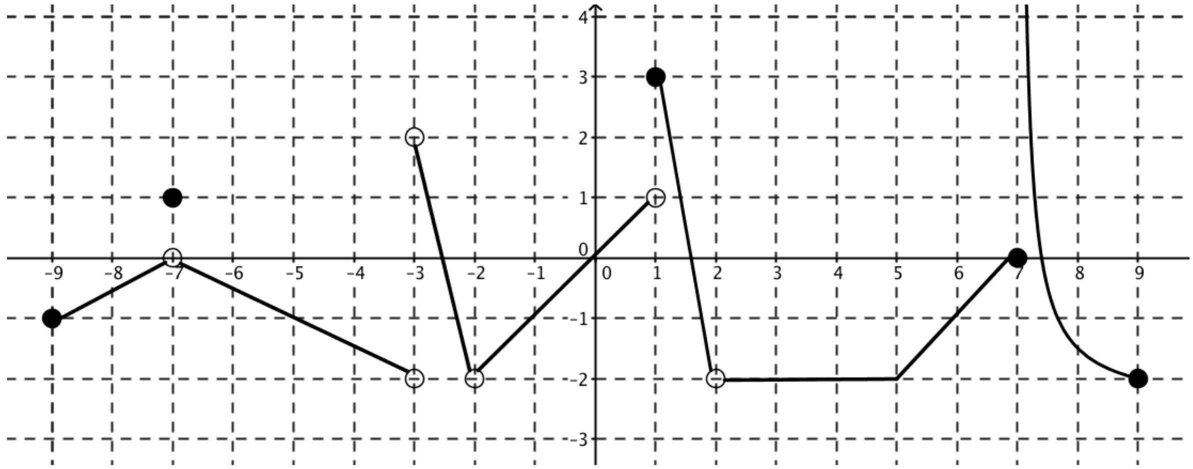


The Sandwich Theorem: If $g(x) \leq f(x) \leq h(x)$ whenever x is sufficiently close to c (but not equal to c) and $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$,

then $\lim_{x\to c} f(x) = L$.

11. If $13 \leq f(x) \leq x^2 - 8x + 29$, determine $\lim_{x\to 4} f(x)$.

1. Let's go back this graph of $y = f(x)$ from Section 2.2:



Focus on $x = -3$.

(a) Review: $\lim_{x \to -3} f(x)$

(b) What value are the y-values on the graph approaching, as the x values get closer and closer to -3 but stay to the left of -3?

(c) What value are the y-values on the graph approaching, as the x values get closer and closer to -3 but stay to the right of -3?

- To determine $\lim_{x \to c^-} f(x)$, you only look at x-values <u>less than</u> c (ones to the <u>left of</u> c on the number line) and see what the corresponding $f(x)$ values are doing as the x-values get closer and closer to c while staying less than c.
  - ⋆ This limit is called a <u>left-hand</u> limit.
  - ⋆ in words: the limit as x approaches c from the <u>left</u>, of $f(x)$
- To determine $\lim_{x \to c^+} f(x)$, you only look at x-values <u>greater than</u> c (ones to the <u>right of</u> c on the number line) and see what the corresponding $f(x)$ values are doing as the x-values get closer and closer to c while staying greater than c.
  - ⋆ This limit is called a <u>right-hand</u> limit.
  - ⋆ in words: the limit as x approaches a from the <u>right</u>, of $f(x)$
- In contrast, $\lim_{x \to c} f(x)$ is called a <u>two-sided</u> limit.
- In order for $\lim_{x \to c} f(x)$ to exist, the limits $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ must be ____________!

1. Using the graph on the previous page, determine the following one-sided limits.

$$\lim_{x\to 9^-} f(x) \qquad \lim_{x\to 7^+} f(x) \qquad \lim_{x\to 7^-} f(x) \qquad \lim_{x\to 1^+} f(x)$$

<u>Note</u> All of the limit laws also apply to one-sided limits! (Be careful, though.)

2. $\lim_{x\to -3^+} \sqrt{x+3}$

3. $\lim_{x\to 3^-} \sqrt{x-3}$

4. $\lim_{x\to 5^-} (x^2+2x+2)\dfrac{x-5}{|x-5|}$

Fact: $\boxed{\lim_{\theta\to 0} \dfrac{\sin(\theta)}{\theta} = 1 \text{ (Remember } \theta \text{ is in radians.)}}$

5. $\lim_{x\to 0} \dfrac{\sin(3x)}{3x}$

6. $\lim_{x\to 0} \dfrac{\sin(4x)}{5x}$

<u>Trig review time:</u> Why does $\lim_{\theta\to 0} \dfrac{\sin(\theta)}{\theta} = 1$?