MATH 3200 - Royal
PRACTICE Exam # 3
Fall 2019

Name (Print): Solutions

Friendly reminders:

- You may not leave the room during the exam period. (If you have a medical issue, let me know.)
- Smart watches, cell phones, and other personal electronic devices must not be on your person and must be stored.
- You don't have to work through the test in order. Go in the order you want to.
- I hope you do a great job!

Problem	Score	Out of
1		25
2		25
3		15
4		15
5		20
Total		100

I will be academically honest in all my academic work and will not tolerate academic dishonesty of others.

Signed: ______________________ Date: __________

1. (5 points each part)

(a) Let $\sim$ be a relation on a set X. Define: $\sim$ is symmetric.

$\forall x, y \in X$, if $x \sim y$ then $y \sim x$

(b) Let $\sim$ be a relation on a set X. Define: $\sim$ is transitive.

$\forall x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$

(c) Define: x is a rational number.

$x = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ with $b \neq 0$

(d) Suppose f is a relation on X and Y. Define: f is a function.

1) $\forall x \in X$, $\exists y \in Y$ such that $(x,y) \in f$, and
2) if $(x,y) \in f$ and $(x,z) \in f$, then $y = z$

(e) Suppose $f : X \to Y$ is a function and $B \subseteq Y$. Define: the preimage of B.

$f^{-1}[B] = \{x \in X \mid f(x) \in B\}$

2. (5 points each) For this problem, refer to the set $S = \{a, b, c\}$.

(a) Give an example of an equivalence relation on S.

$\{(a,a),(b,b),(c,c)\}$

(b) Give an example of a relation on S that is reflexive and is symmetric, but is not transitive, or explain why it is impossible to give such a relation.

$\{(a,a),(b,b),(c,c),(a,b),(b,a),(b,c),(c,b)\}$

(c) Give an example of a relation on S that is not reflexive, but is symmetric and is transitive, or explain why it is impossible to give such a relation.

$\{(a,a)\}$

(d) The relation $\mathcal{R} = \{(a, b), (b, a)\}$ is a relation on S. Is it symmetric? Explain briefly.

Yes: if $(x,y) \in \mathcal{R}$, then $(y,x) \in \mathcal{R}$ as well

(e) The relation $\mathcal{T} = \{(a, a)\}$ is a relation on S. Is it an equivalence relation? Explain briefly.

no: it's not reflexive

3. (15 points) Let A and B be sets. Use Axiom 4.24 to prove that

$$(A \cap B)^c = A^c \cup B^c$$

<u>Proof</u> We use axiom 4.24. We first show that $(A \cap B)^c \subseteq A^c \cup B^c$.

Let $x \in (A \cap B)^c$. Then $x \notin A \cap B$, so $x \notin A$ or $x \notin B$. As a result, $x \in A^c$ or $x \in B^c$, so $x \in A^c \cup B^c$. Thus

$$(A \cap B)^c \subseteq A^c \cup B^c.$$

We next show that $A^c \cup B^c \subseteq (A \cap B)^c$.

Let $y \in A^c \cup B^c$; then $y \in A^c$ or $y \in B^c$. Assume $y \in A^c$; then $y \notin A$, so $y \notin A \cap B$. Thus $y \in (A \cap B)^c$. Similarly, $y \in B^c$ implies $y \in (A \cap B)^c$. Thus

$$A^c \cup B^c \subseteq (A \cap B)^c.$$

$\blacksquare$

4. (15 points) Let B be a set, and let $S = \mathcal{P}(B)$, the power set of B. Define a relation $\sim$ on S by $X \sim Y$ if $X \subseteq Y$. Prove that $\sim$ is a partial ordering on S.

Proof We will show that $\sim$ is reflexive, antisymmetric, and transitive.
We first show $\sim$ is reflexive. Let $X \in S$; then $X \in \mathcal{P}(B)$; in particular, X is a set, so $X \subseteq X$. Thus $X \sim X$.

We next show that $\sim$ is antisymmetric. Let $X, Y \in S$, and assume $X \sim Y$ and $Y \sim X$. Then $X \subseteq Y$ and $Y \subseteq X$, so $X = Y$ by Axiom 4.24. Thus $\sim$ is antisymmetric.

Finally, we show that $\sim$ is transitive. Let $X, Y, Z \in S$, and suppose $X \sim Y$ and $Y \sim Z$. Then $X \subseteq Y$ and $Y \subseteq Z$, so $X \subseteq Z$ by the transitive property for subsets. Thus $X \sim Z$, so $\sim$ is transitive.

$\blacksquare$

5. (20 points) Let $S = \mathbb{Z}$, and define the relation $\sim$ on S by $a \sim b$ if $3|(a-b)$. Show that $\sim$ is an equivalence relation.

<u>Proof</u> We first show $\sim$ is reflexive.
Let $x \in \mathbb{Z}$; since $x - x = 0$ and $3|0$, it follows that $3|(x-x)$, so $x \sim x$. Thus $\sim$ is reflexive.

We next show $\sim$ is symmetric. Let $x, y \in \mathbb{Z}$, and assume $x \sim y$. Then $3|(x-y)$, so
$x - y = 3j$ for some $j \in \mathbb{Z}$. As a result,
$y - x = 3(-j)$. Since $-j \in \mathbb{Z}$, it follows that
$3|(y-x)$, so $y \sim x$; thus $\sim$ is symmetric.

Finally we show $\sim$ is transitive. Let $x, y, z \in \mathbb{Z}$, and assume $x \sim y$ and $y \sim z$. Then
$x - y = 3k$ and $y - z = 3\ell$ where $k, \ell \in \mathbb{Z}$.
As a result, $x - z = (x-y)+(y-z) = 3k + 3\ell = 3(k+\ell)$.
Since $k + \ell \in \mathbb{Z}$, it follows that $3|(x-z)$, so
$x \sim z$. Thus $\sim$ is transitive.

$\blacksquare$

Note: This relation is very important in MATH 4000/6000. We say that $a \cong b \pmod{n}$ (read aloud as "a is congruent to b mod n") if $n|(a-b)$. In this problem, you are proving that congruence mod 3 is an equivalence relation.

↳ in class, we looked at congruence mod 5 and congruence mod 2

Here is an extra page for scratch work. Make a note on the problem page if you want me to look at your work here.