<u>The Mean Value Theorem</u> Let f be a function such that

1. f is continuous on $[a, b]$ and
2. f is differentiable on (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

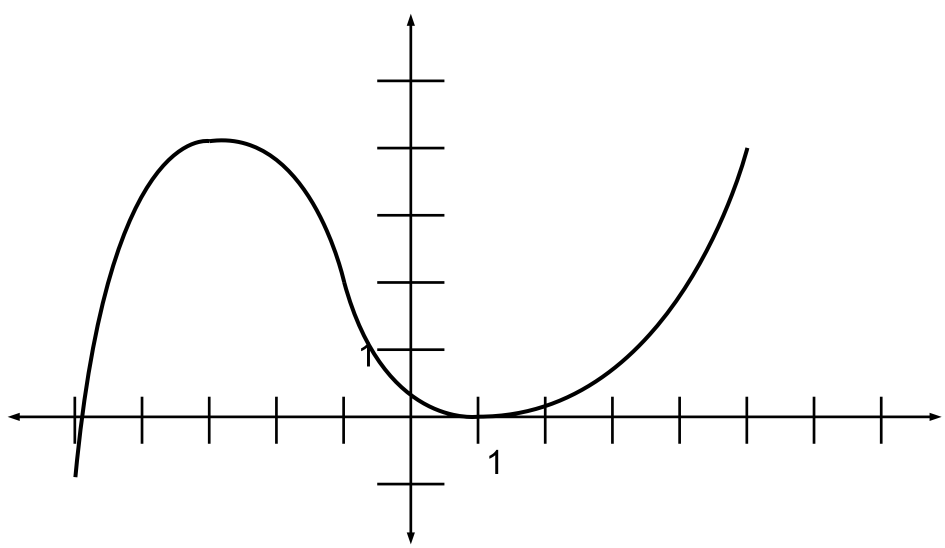
<u>Important Observations.</u>

$$f'(c) = \text{slope of } \textit{tangent} \text{ line to the graph of } f \text{ at } (c, f(c))$$

$$\frac{f(b) - f(a)}{b - a} = \text{slope of } \textit{secant} \text{ line to the graph of } f \text{ thru points } (a, f(a)) \text{ and } (b, f(b))$$

So, the Mean Value Theorem says that there's a point on your graph at which the tangent slope is the same thing as the secant slope through the endpoints of your graph.

1. The graph below is the graph of $y = f(x)$. Does the mean value theorem apply to this function on the interval $[-5, 5]$? If so, estimate all values of c satisfying the conclusion of the Mean Value Theorem on the interval.



2. Does $f(x) = \begin{cases} 1 & \text{if } x > 3 \\ 0 & \text{if } x \leq 3 \end{cases}$ satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 4]$? Why or Why not?

3. Consider the function $f(x) = \frac{1}{3}x^3 - 5x - 4$.

   (a) Verify that $f(x)$ satisfies the hypotheses of the Mean Value Theorem on $[-1, 5]$.

   (b) Determine the slope of the secant line through $(-1, f(-1))$ and $(5, f(5))$.

   (c) Find a number $c \in (-1, 5)$ such that the slope of the tangent line at $(c, f(c))$ is equal to the slope of the secant line through $(-1, f(-1))$ and $(5, f(5))$.

4. Determine the average rate of change of the function $f(x) = |x|$ on the interval $[-1, 2]$. Does the mean value theorem apply to $f(x) = |x|$? If so, determine a value of c in $(-1, 1)$ such that $f'(c)$ is equal to this average slope.

5. Does $f(x) = \sqrt{x - 2}$ satisfy the hypotheses of the Mean Value Theorem on the interval $[2, 4]$? Why or why not?

6. Does $f(x) = x^{2/3} - 4x$ satisfy the hypotheses of the Mean Value Theorem on the interval $[-1, 1]$? Why or why not?

Applications of the Mean Value Theorem Both of these applications assume that f is continuous on $[a, b]$ and differentiable on (a, b).

Zero Derivative Property If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

1. Use the Mean Value Theorem to explain why the Zero Derivative Property must be true.

Matching Derivative Property If $f'(x) = g'(x)$ for all x in an interval (a, b), then f and g differ by a constant on (a, b); that is, there exists a constant c such that $f(x) = g(x) + c$ for all x in (a, b).

2. Use the Zero Derivative Property to explain why the Matching Derivative Property must be true.

3. What is the significance of the Matching Derivative Property?

4. Determine a function $f(x)$ satisfying $f'(x) = 13x + 5$ and $f(1) = 3/2$.

5. Determine a function $f(x)$ satisfying $f''(x) = 9x^2 - 27$ and $f(1) = 1/4$, $f'(1) = 12$.

6. Determine a function $f(x)$ satisfying $f'(x) = \sin(8x)$ such that the point $(0, 5/8)$ is on the graph of $f(x)$.

7. Proving an identity for inverse trig functions.

   (a) Review: What is $\frac{d}{dx}(\cos^{-1}(x))$?

   (b) Review: What is $\frac{d}{dx}(-\sin^{-1}(x))$?

   (c) Based on your answers above, what do you conclude about $\cos^{-1}(x)$ and $\sin^{-1}(x)$? Be as specific as possible.