Royal problems

R1. Let P: 15 is odd, and Q: 2 is even. State each of the following in words, and determine whether it is true or false.

(a) $P \wedge Q$

(b) $P \wedge (\sim Q)$

(c) $(\sim P) \vee Q$

(d) $(\sim P) \vee (\sim Q)$

R2. Make a truth table for the compound statement $B \wedge (\sim B)$. Give an example of a statement B, and interpret the compound statement $B \wedge (\sim B)$ for your statement B. Then make a truth table for $B \vee (\sim B)$, and interpret the compound statement for your statement B.

R3. Write a *useful* negation for each of the following statements:

(a) All cows eat grass.

(b) There is a horse that does not eat grass.

(c) Some cows are spotted.

(d) No car has 15 cylinders.

R4. Suppose your parent or caregiver told you as a child, "If we move to a house with enough land, we will get you a horse." Under each of the following scenarios, determine whether your parents were truthful (i.e. whether the statement above was true):

- Your family never moves to a house with enough land, but they figure out a way to get you a horse anyway.
- Your family never moves to a house with enough land. You go off to college, disappointed and without a horse.
- Your family moves to a house with enough land, and your parents do not get you a horse.
- Your family moves to a house with enough land, and your parents get you a horse.

R5. Determine whether the following biconditional statements are true or false:

- The number two is even if and only if the number 10 is odd.
- Hendon is your instructor for MATH 3200 if and only if this class is 50 minutes long.
- The capital of Tennessee is Atlanta if and only if the capital of Georgia is Montgomery.
- Kirby Smart is UGA's football coach if and only if Nick Saban is the University of Alabama's football coach.

R6. Negate the following statements:

(a) Every rose has its thorn.

(b) If you don't eat your meat, you can't have any pudding.

(c) Big girls don't cry.

(d) Sometimes love just ain't enough.

R7. Determine the truth value of the statement. Then determine the negation of the statement.

(a) The number π is a rational number.
(b) There exists a real number x satisfying $x^2 + 1 < 0$.
(c) There exist two integers x and y satisfying $xy = -1/2$.
(d) For every real number x, we have $x^2 + 1 > 1$.
(e) If x is a real number, then $x^2 + 1 > 0$.
(f) If x is an odd integer, then $x - 2$ is an odd integer.

R8. Determine the truth value of the statement. Then determine the negation of the statement. (See Houston p. 87-88.)

(a) If n is an integer, then n is even or n is odd.
(b) If n is an integer, then $n^2 \geq 0$ and $n^2 \geq n$.
(c) If n is an integer, then $2^{n-1} < 2^n < 2^{n+1}$.
(d) For every even integer x, there exist odd integers y and z such that $x = y + z$.
(e) There exists an integer x such that for every real number y, $x \cdot y = y$.

R9. Prove: For any odd integer n, the integer $5n^2 + 2n$ is odd.

R10. Prove: Let n be an integer. If $7n + 2$ is even, then n is even.

Rules for inequalities.
Assume a, b, and c are real numbers. (Rule 1) If $a > b$ and $b > c$, then $a > c$. (Rule 2) If $a > b$ then $a + c > b + c$. (Rule 3) If $a > b$ and $c > 0$, then $ac > bc$. (Rule 4) If $a > b$ and $c < 0$, then $ac < bc$. (Rule 5) For every real number a, exactly one of the following is true: $a > 0$ $\qquad$ $a = 0$ $\qquad$ $a < 0$

R11. Assume x and y are real numbers. Prove, using the inequality rules above: if $x > 0$ and $y < 0$ then $xy < 0$.

R12. Let $x \in \mathbb{R}$. Prove, using the inequality rules above: if $x \neq 0$, then $x^2 > 0$.

R13. Let $x \in \mathbb{R}$. Prove, using the inequality rules above: $x^2 \geq 0$ for every $x \in \mathbb{R}$.

R14. Let $x \in \mathbb{R}$. Prove, using the inequality rules above: $x > 0$ implies $\dfrac{1}{x} > 0$.

R15. Let $x \in \mathbb{R}$. Prove, using the inequality rules above: if $x < 1$, then $2x - 1 < 5$.

R16. Let x be a real number. Prove, using the inequality rules above: if $x < 6$, then $2x - 1 < 13$

R17. Let x be a real number. Prove, using the inequality rules above: if $x > 3$ then $4x + 7 > 16$.

R18. Let a, b, c, x, and y be integers, with $a \neq 0$. Prove: if $a|b$ and $a|c$, then $a|(bx + cy)$.

R19. Consider the open sentence $P(n) : 9 + 13 + \ldots + (4n + 5) = \dfrac{4n^2 + 14n + 1}{2}$, where $n \in \mathbb{N}$.

(a) Verify the implication $P(k) \implies P(k+1)$ for an arbitrary positive integer k.

(b) Is $\forall n \in \mathbb{N}, P(n)$ true?

R20. Use induction to prove that $3|(4^n - 1)$ for every natural number n.

R21. Use induction to prove that 8 divides $9^n - 1$ for every natural number n.

R22. Use induction to prove that $8|(5^{2n} - 1)$ for every natural number n.

R23. Use induction to prove that $3^{n+3} > (n+3)^3$ for every natural number n.

R24. Use induction to prove that $2^n > n^3$ for every integer $n \geq 10$.

R25. Use induction to prove that $2! \cdot 4! \cdot 6! \cdot \ldots \cdot (2n)! \geq [(n+1)!]^n$ for every positive integer n.

R26. Use induction to prove that, for every natural number n, $\frac{d}{dx}(x^n) = nx^{n-1}$. You may assume the differentiation formulas: $\frac{d}{dx}(x) = 1$ and $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$.

R27. A sequence is defined recursively by $a_1 = 1$, $a_2 = 2$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Conjecture a formula for a_n and verify your conjecture is correct.

R28. Let $a_1 = 0$, $a_2 = 1$, and for all integers $n \geq 3$ define $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$. Show that $a_n = \frac{2^{n-1} + (-1)^n}{3 \cdot 2^{n-2}}$ for all $n \in \mathbb{N}$.

R29. Let a_n be the sequence defined by $a_1 = 1$, $a_2 = 8$, and $a_n = a_{n-1} + 2a_{n-2}$ for any integer $n \geq 3$. Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for every natural number n.

R30. Let A be a subset of a universal set U. Prove that $(A^c)^c = A$.

R31. For sets A and B, find a necessary and sufficient condition for $(A \times B) \cap (B \times A) = \emptyset$. Verify that the condition is necessary and sufficient.

Hint: Your goal is to complete the sentence $(A \times B) \cap (B \times A) = \emptyset$ if and only if ... (and prove it).

R32. For sets $A = \{a, b, c\}$ and $B = \{x, y, z\}$:

(a) Determine the set product $A \times B$.

(b) Determine the set product $B \times A$.

(c) Is $A \times B$ equal to $B \times A$? Why or why not?

R33. (a) For the set $A = \{\{1, 2\}, 3, 4\}$, determine $\mathcal{P}(A)$.

(b) For the set $B = \{\emptyset, 1, 2\}$, determine $\mathcal{P}(B)$.

R34 For $S = \{1, 2, 3\}$ and $T = \{a, b, c, d\}$, give an example of a relation $\mathcal{R}$ on S and T. Then find the domain, codomain, and range of $\mathcal{R}$.

R35 For $S = \{1, 3, 5\}$ and $T = \{2, 4, 11\}$, define $\mathcal{R}$ on S and T by $(a, b) \in \mathcal{R}$ if $2a > b$. Write out the relation $\mathcal{R}$ using set notation. Then find the domain, codomain, and range of $\mathcal{R}$.

R36 (modified Excercise 5.3): In an imaginary classroom on the first day of class, there are 3 students. The following information describes students' knowledge of other students' names (and no one else knows anyone else's name):
Abril knows her own name.
Barack knows his own name.
Chris knows their own name.
Abril knows Barack's name.
Abril knows Chris's name.
Chris knows Barack's name.
Chris knows Abril's name.
Barack knows Abril's name.

$S = \{\text{Abril}, \text{Barack}, \text{Chris}\}$. Define a relation $\mathcal{R}$ on S by $x\mathcal{R}y$ if x knows y's name. Write out the relation $\mathcal{R}$ using set notation.

R36 Consider the relation $\mathcal{R}$ in example R35. Is is reflexive? Symmetric? Transitive?