

# Calculus

***Calculus*** - the study of change, as related to functions

Formally co-developed around the 1660's by Newton and Leibniz

Two main branches - differential and integral

Central role in much of modern science

Physics, especially kinematics and electrodynamics

Economics, engineering, medicine, chemistry, *etc.*

# Integrals

We do integrals by *recognizing* the integrand as the derivative of a known function

$$\int c f(x) dx = c \int f(x) dx$$

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

$$\int k dx = kx + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int \frac{1}{x^2 + 1} dx = \tan^{-1}x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}x + C$$

# Integration Techniques

Main strategy for integration

① Know your derivatives forward and backward! ①

Unfortunately, this severely limits the number of integrals we can do

If ①   ① fails, our options are limited!

②: Look clever substitutions to get back to ①   ①

③: Look for special forms to get back to ①   ①

④: Exploit the symmetry of the definite integral

# Integration - Derivative Substitutions

Consider the following integrals

$$I_1 = \int \frac{1}{1+x} dx$$

$$I_2 = \int \frac{1}{1+x^2} dx$$

# Integration - Derivative Substitutions

Recall the chain rule

$$\frac{d}{dx} [F(g(x))] = \frac{dF}{dx} \Big|_{x=g(x)} g'(x) = f(g(x))g'(x)$$

Integrating the LHS and the RHS we get

$$F(g(x)) = \int f[g(x)]g'(x)dx$$

Suggests a method of solving integrals of this form, provided we have a suitable antiderivative

In general, we 'transform' the integral via a clever substitution

If  $\boxed{\textcircled{1} u = g(x)}$   $\Rightarrow$   $\boxed{\textcircled{2} \frac{du}{dx} = g'(x)}$  then

$$\int f[g(x)]g'(x)dx = \int f(u)\frac{du}{dx}dx = \int f(u)du$$

If the antiderivative of  $f$  is known, the problem is (essentially) solved

# Derivative Substitutions - Examples

Example 1: Calculate

$$\int \frac{x}{1+x^2} dx$$

# Derivative Substitutions - Examples

Example 2: Calculate

$$\int \tan \theta \, d\theta$$

# Derivative Substitutions - Examples

Example 3: Calculate

$$\int t^2 \csc(t^3) \cot(t^3) dt$$



# Derivative Substitutions - Examples

Example 4: Calculate

$$\int e^{x/5} dx$$

# Derivative Substitutions - Definite Integrals

If we are facing a definite integral of the form

$$I = \int_a^b f[g(x)]g'(x)dx$$

the previous procedure requires an extra step to transform the limits

If ①  $u = g(x)$   $\Rightarrow$  ②  $\frac{du}{dx} = g'(x)$  and ③  $u_1 = g(a), u_2 = g(b)$  then

$$\int_a^b f[g(x)]g'(x)dx = \int_a^b f(u)\frac{du}{dx}dx = \int_{u_1}^{u_2} f(u)du = F(u_2) - F(u_1) = I$$

# Derivative Substitutions - Examples

Example 5: Evaluate

$$\int_1^{e^2} \frac{\ln x}{x} dx$$

# Integration via Parts

We have seen how to integrate when

① the integrand has a known antiderivative

② the integrand contains a function and its derivative

What about

$$\int x \sin x \, dx \quad \text{or} \quad \int e^t \sin t \, dt$$

# Integration by Parts - Motivation

Last time, we looked at the chain rule for inspiration.

This time, let's look at the product rule!

$$\frac{d}{dx}[fg] = \frac{df}{dx}g + f\frac{dg}{dx}$$

$$\int f \frac{dg}{dx} dx = fg - \int g \frac{df}{dx} dx$$

# Integration by Parts - Examples

Example 1: Find

$$\int x \sin x \, dx$$

# Integration by Parts - Examples

Example 2: Find

$$\int x^3 \sin x \, dx$$

# Integration by Parts - Examples

Example 3: Find

$$\int x^3 e^{4x^2} dx$$



# Integration by Parts - Examples

Example 4: Find

$$\int \log t \, dt$$

# Integration by Parts - Examples

Example 5: Evaluate

$$\int_0^1 \frac{x^2 + 1}{e^x} dx$$

# Integration by Parts - Examples

Example 6: Find

$$\int e^t \sin 2t \, dt$$

# Integration via Partial Fractions

When an integrand of the form  $\frac{P(x)}{Q(x)}$  with polynomials  $P(x)$  and  $Q(x)$

① if  $P(x) = cQ'(x)$  then we can use a derivative substitution

② else, we have to be more clever!

# Partial Fractions - Aside

Consider the following integral

$$\int \left[ \frac{1}{x-1} - \frac{1}{x} \right] dx$$

# Partial Fractions - Procedure - $n < m$

If we consider

$$\int \frac{P(x)}{Q(x)} dx$$

with  $\deg(P) = n$ ,  $\deg(Q) = m$  and  $n < m$

- ① Factor  $Q(x) = q_1(x)q_2(x) \dots q_m(x)$  with  $\deg(q_i) = d_i$
- ② Write  $\frac{P(x)}{Q(x)} = \frac{p_1(x)}{q_1(x)} + \frac{p_2(x)}{q_2(x)} + \dots + \frac{p_m(x)}{q_m(x)}$  with  $\deg(p_i) < d_i$
- ③ Using the equality in ②, solve for each of the  $p_i(x)$
- ④ Integrate each of the individual partial fractions

# Partial Fractions

Denominator - $q_i(x)$	Partial Fraction - $\frac{p_i(x)}{q_i(x)}$
$(x - a)$	$\frac{A}{x-a}$
$(x - a)^r$	$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_r}{(x-a)^r}$
$(ax^2 + bx + c)$	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2 + bx + c)^r$	$\frac{A_1x+B_1}{(ax^2+bx+c)} + \dots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$

# Partial Fractions - Examples

Example 1: Find

$$\int \frac{y}{2y^2 - 3y + 1} dy$$



# Partial Fractions - Examples

Example 2: Find

$$\int \frac{10}{5x^2 - 2x^3} dx$$

# Partial Fractions - Examples

Example 3: Find

$$\int \frac{10}{(x-1)(x^2+9)} dx$$

# Partial Fractions - Procedure - $n \geq m$

If we consider

$$\int \frac{P(x)}{Q(x)} dx$$

with  $\deg(P) = n$ ,  $\deg(Q) = m$  and  $n \geq m$  we add an extra step!

- ① Long division  $\Rightarrow P(x) = g(x) + \frac{R(x)}{Q(x)}$  with  $\deg(R) < \deg(Q)$
- ① Factor  $Q(x) = q_1(x)q_2(x) \dots q_m(x)$  with  $\deg(q_i) = d_i$
- ② Write  $\frac{R(x)}{Q(x)} = \frac{r_1(x)}{q_1(x)} + \frac{r_2(x)}{q_2(x)} + \dots + \frac{r_m(x)}{q_m(x)}$  with  $\deg(r_i) < d_i$
- ③ Using the equality in ②, solve for each of the  $r_i(x)$
- ④ Integrate  $g(x)$  and each of the individual partial fractions

# Partial Fractions - Examples

Example 4: Find

$$\int_3^4 \frac{x^3 - 2x^2 - 4}{x^3 - 2x^2} dx$$

# Partial Fractions - Examples

Example 5: Find

$$\int \frac{x^5 + x - 1}{x^3 + 1} dx$$

# Partial Fractions - Examples

Example 6: Find

$$\int \frac{3t^2 + t + 4}{t^3 + t} dx$$

# Partial Fractions - Examples

Example 7: Find

$$\int \frac{1}{1 + e^x} dx$$

# Integration of Trigonometric Functions

Our integration repertoire now includes integrals where

- ① the integrand has a known antiderivative
- ② the integrand contains a function and its derivative
- ③ the integrand is a ‘special’ product of functions

$$\int \sin \theta \cos \theta \, d\theta$$



# Trigonometric Functions - Using Identities

Often times, we can use trigonometric identities to reformulate an integral

$$\int \sin \theta \cos \theta \, d\theta$$

# Trigonometric Functions - Using Identities

Odd powers of trigonometric functions

$$\textcircled{1} \int \cos^3 t \, dt$$

$$\textcircled{2} \int \cos^5 t \, dt$$

# Trigonometric Functions - Using Identities

What about even powers?

$$\textcircled{3} \int \cos^2 t \, dt$$

# Trigonometric Functions - Using Identities

What about even powers?

$$\textcircled{4} \int \cos^4 t \, dt$$

# Trigonometric Functions - Using Identities

Similar strategies apply for integrals involving  $\tan \theta$  and  $\sec \theta$

Use  $1 + \tan^2 \theta = \sec^2 \theta$ ,  $\frac{d}{d\theta} \tan \theta = \sec^2 \theta$  and  $\frac{d}{d\theta} \sec \theta = \sec \theta \tan \theta$

$$\textcircled{4} \int \tan^5 t \sec^3 t \, dt$$

# Trigonometric Functions - Examples

Example 1: Find

$$\int \cos^2 x \sin^2 x \, dx$$

# Trigonometric Functions - Examples

Example 2: Find

$$\int \tan^2 \theta \, d\theta$$

# Trigonometric Functions - Examples

Example 3: Find

$$\int \frac{1}{1 - \sin \theta} d\theta$$



# Trigonometric Functions - Examples

Example 4: Find

$$\int_0^{\pi/4} \sqrt{1 - \cos 4\theta} \, d\theta$$

# Integration via Trigonometric Substitution

We have strategies for integrals involving

- ① known antiderivatives      and      ② derivative substitutions

Unfortunately, unlike differentiation, integration is ‘*unstable*’

Small changes to the integrand can lead to difficult integrals

Need to develop new techniques!

# Trigonometric Substitutions - Methods

Consider integrals that contain the following forms and their powers

$$\sqrt{k^2 - x^2}, \quad \sqrt{x^2 + k^2} \quad \text{and} \quad \sqrt{x^2 - k^2}, \text{ with } k > 0$$

We look for substitutions that simplify such integrals

① Right triangles to extract the substitution

② Trigonometric identities

# Trigonometric Substitutions - Triangles

The 'Pythagorean' form of the integrand suggests the related right triangle

Integrand	Triangle
$\sqrt{x^2 + k^2}, \frac{1}{\sqrt{x^2+k^2}}, \frac{1}{x^2+k^2}$	
$\sqrt{k^2 - x^2}, \frac{1}{\sqrt{k^2-x^2}}$	
$\sqrt{x^2 - k^2}, \frac{1}{\sqrt{x^2-k^2}}$	

# Trigonometric Substitutions - Identities

Alternatively, when the integrand contains

$$(\sqrt{k^2 - x^2})^{\pm n}, (\sqrt{x^2 - k^2})^{\pm n}, \text{ or } (\sqrt{x^2 + k^2})^{\pm n}, k > 0$$

we can use the various forms of the 'Pythagorean' trig identity to find a suitable substitution

$$\textcircled{1} \quad \cos^2 \theta + \sin^2 \theta = 1$$

$$\textcircled{2} \quad 1 + \tan^2 \theta = \sec^2 \theta$$

# Trigonometric Substitutions - Identities

This suggests the following set of substitutions

Integrand	Substitution
$\sqrt{k^2 - x^2}, \quad \frac{1}{\sqrt{k^2 - x^2}}$	$x = k \sin \theta$
$\sqrt{x^2 + k^2}, \quad \frac{1}{\sqrt{x^2 + k^2}}, \quad \frac{1}{x^2 + k^2}$	$x = k \tan \theta$
$\sqrt{x^2 - k^2}, \quad \frac{1}{\sqrt{x^2 - k^2}}$	$x = k \sec \theta$

# Trigonometric Substitutions - Examples

Example 1: Find

$$\int \frac{1}{\sqrt{9 - x^2}} dx$$

# Trigonometric Substitutions - Examples

Example 2: Find

$$\int \frac{1}{x^2 \sqrt{x^2 + 4}} dx$$



# Trigonometric Substitutions - Examples

Example 3: Find

$$\int \frac{1}{\sqrt{x^2 - 6x}} dx$$

# Trigonometric Substitutions - Examples

Example 4: Evaluate

$$\int_0^{\frac{\sqrt{2}}{2}} \frac{x^2}{(1-x^2)^{3/2}} dx$$