1a

ARCLENGTH OF $x = \frac{1}{3}\sqrt{y}\,(y-3)$ FROM $1 \le y \le 9$

SOL'N: USING $L = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$

WHERE $c = 1$, $d = 9$ AND $x = x(y) = \frac{1}{3} y^{1/2}(y-3)$

$$= \frac{1}{3} y^{3/2} - y^{1/2}$$

THUS $\frac{dx}{dy} = \frac{1}{3}\left(\frac{3}{2}\right) y^{1/2} - \frac{1}{2} y^{-1/2}$

$$= \frac{1}{2} y^{1/2} - \frac{1}{2} y^{-1/2}$$

$$= \frac{1}{2}\left(y^{1/2} - y^{-1/2}\right)$$

THUS $\left(\frac{dx}{dy}\right)^2 = \left[\frac{1}{2}\left(y^{1/2} - y^{-1/2}\right)\right]^2 = \frac{1}{4}\left[y - 2 + y^{-1}\right]$

$$= \frac{y}{4} - \frac{1}{2} + \frac{1}{4y}$$

SO $1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left[\frac{y}{4} - \frac{1}{2} + \frac{1}{4y}\right] = \frac{y}{4} + \frac{1}{2} + \frac{1}{4y}$

$$= \left[\frac{y^{1/2} + y^{-1/2}}{2}\right]^2$$

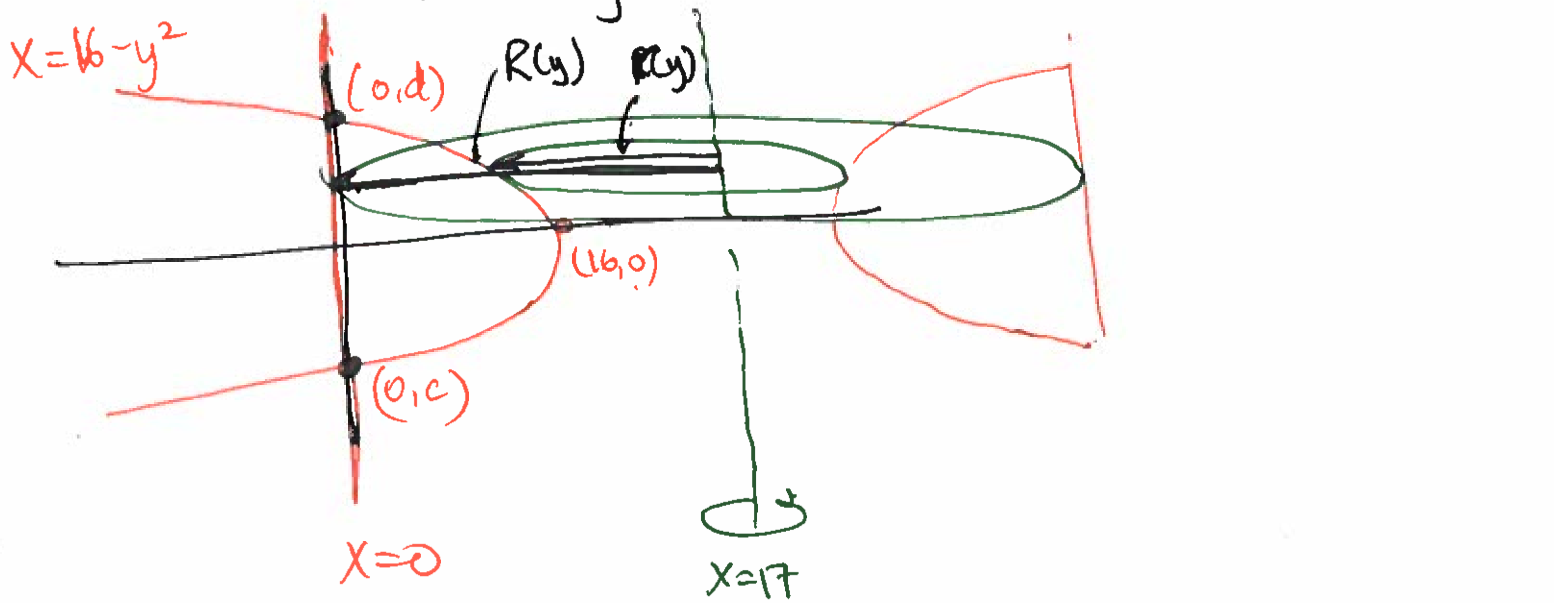
$\therefore$ $L = \int_1^9 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

$$= \int_1^9 \sqrt{\left[\frac{y^{1/2} + y^{-1/2}}{2}\right]^2}\, dy = \frac{1}{2}\int_1^9 \left(y^{1/2} + y^{-1/2}\right) dy$$

$$= \frac{1}{2}\left[\frac{2}{3} y^{3/2} + 2y^{1/2} + k\right]_1^9 = \boxed{\frac{32}{3}}$$

1d) VOLUME FOR REGION BOUNDED BY

$x = 16 - y^2$ AND $x = 0$ ROTATED ABOUT $x = 17$



SLICING $\perp$ TO AXIS OF ROTATION $\rightarrow$ WASHERS!
INTEGRATED WRT y

$$V = \int_c^d \pi R^2(y)\,dy - \int_c^d \pi r^2(y)\,dy$$

WHERE $c + d$ ARE THE INTERSECTION BETWEEN ① $x = 16 - y^2$ ② $x = 0$

SO $0 = 16 - y^2 \rightarrow y^2 = 16 \rightarrow y = \pm 4$

$\boxed{c = -4}$, $\boxed{d = 4}$

AND $R(y) = 17 - 0 = 17$

$r(y) = 17 - (16 - y^2) = 1 + y^2$

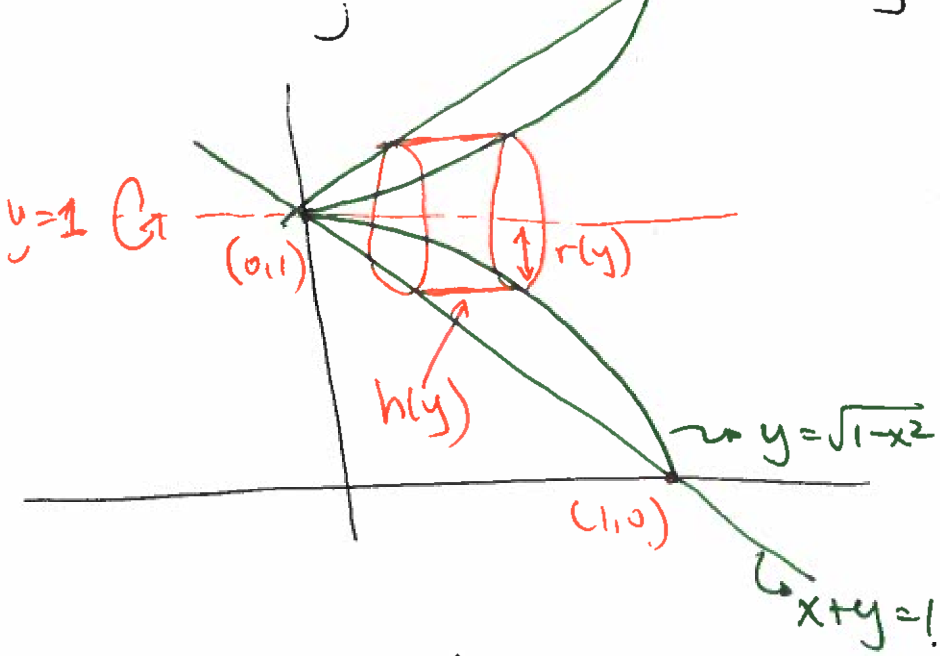
SO $$V = -\int_{-4}^{4} \pi (1+y^2)^2\,dy + \int_{-4}^{4} \pi [17]^2\,dy = -\pi \int_{-4}^{4} (1 + 2y^2 + y^4)\,dy + \pi \int_{-4}^{4} 289\,dy$$

$$= \pi \left[-y - \frac{2}{3}y^3 - \frac{1}{5}y^5 + 289y + K \right]_{-4}^{4}$$

$$= 6784\pi/3$$

2b) VOLUME FOR REGION BOUNDED BY

$y = \sqrt{1-x^2}$ AND $x+y=1$ ROTATED ABOUT $y=1$



SLICING ∥ TO AXIS OF ROTATION

↳ CYLINDRICAL SHELLS

[INTEGRATING WRT y]

$$V = \int_c^d 2\pi\, r(y) \cdot h(y)\, dy$$

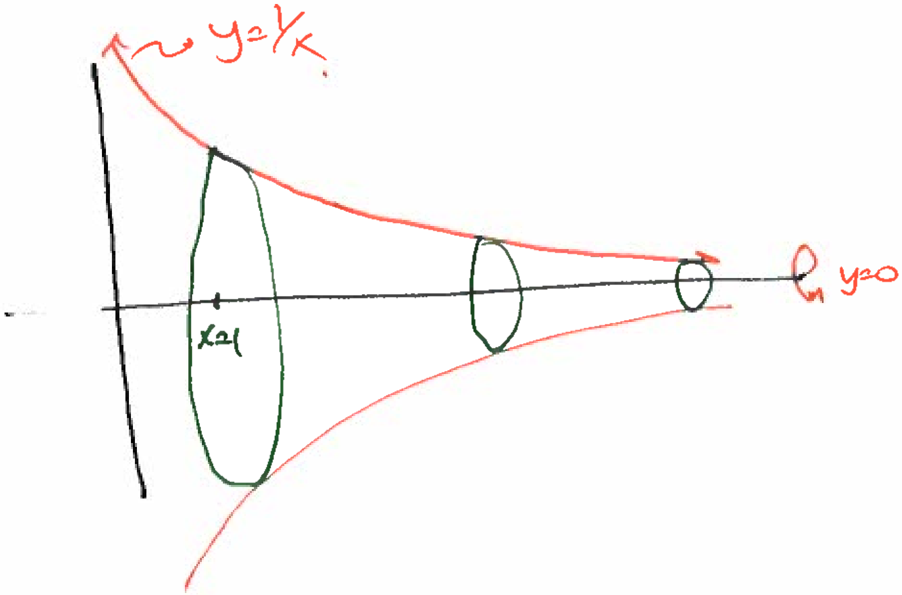
WHERE
$c = 0$
$d = 1$

$$r(y) = 1 - y \qquad h(y) = x_2(y) - x_1(y)$$
$$= \sqrt{1-y^2} - (1-y)$$

so
$$V = 2\pi \int_0^1 (1-y) \cdot \left[\sqrt{1-y^2} - (1-y)\right] dy$$

$$= 2\pi \left[\underbrace{\int_0^1 \sqrt{1-y^2}\, dy}_{A} - \underbrace{\int_0^1 y\sqrt{1-y^2}\, dy}_{B} - \underbrace{\int_0^1 (1-y)^2\, dy}_{C} \right]$$

③ PORTION OF THE CURVE $y = 1/x$ FOR $1 \leq x < \infty$ ROTATED ABOUT $y=0$



[RESULTING S.O.R IS KNOWN AS GABRIEL'S HORN

OR TORRICELLI'S TRUMPET]

a) FOR THE ARCLENGTH $L = \int_1^{\infty} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

WHERE $\frac{dy}{dx} = -1/x^2$ SO $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{1}{x^4}}$

AND $L = \int_1^{\infty} \sqrt{1 + \frac{1}{x^4}}\, dx = \int_1^{\infty} \frac{\sqrt{x^4+1}}{x^2}\, dx$

THE INTEGRAL ABOVE CANNOT BE EVALUATED DIRECTLY BY USING ELEMENTARY FUNCTIONS, HOWEVER NOTE THAT:

$1 < 1 + 1/x^4 \longrightarrow \sqrt{1} < \sqrt{1 + 1/x^4}$ OR $\int_1^{\infty} 1\, dx < \int_1^{\infty} \sqrt{1 + 1/x^4}\, dx$

[SINCE $\frac{1}{x^4} > 0$]

BUT $\int_1^{\infty} 1\, dx = \int_1^{\infty} x^0\, dx$ ↳ DIVERGES SINCE $\int_a^{\infty} x^{-p}\, dx$ DIVERGES WHEN $p \leq 1$

3b

FOR S.A. OF REVOLUTION

$$S.A = \int_1^\infty 2\pi\, r(x) \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^\infty 2\pi \frac{1}{x} \sqrt{1+\frac{1}{x^4}}\, dx$$

AGAIN, DIRECT INTEGRATION IS IMPOSSIBLE USING ELEMENTARY FUNCTIONS

HOWEVER NOTE THAT $1 < 1+\frac{1}{x^4}$ [SINCE $\frac{1}{x^4}>0$]

so $\sqrt{1} < \sqrt{1+1/x^4}$

$1 < \sqrt{1+1/x^4}$

$1/x < \frac{1}{x}\sqrt{1+1/x^4}$

AND $\int_1^\infty \frac{1}{x}\, dx < \int_1^\infty \frac{1}{x}\sqrt{1+1/x^4}\, dx$

BUT $\int_1^\infty \frac{1}{x}\, dx$ DIVERGES SINCE $\int_a^\infty x^{-p}\, dx$ DIVERGES IF $p \le 1$

SO BY COMPARISON $\int_1^\infty \frac{1}{x}\sqrt{1+1/x^4}\, dx$ ALSO DIVERGES.

3c

FOR THE VOLUME OF THE S.O.R.

SLICING $\perp$ TO AXIS OF ROTATION $\rightarrow$ DISKS [INTEGRATING WRT x]

$$V = \int_1^\infty \pi [R(x)]^2 \, dx \quad \text{WHERE} \quad R(x) = \frac{1}{x}$$

so

$$= \int_1^\infty \pi \left[\frac{1}{x}\right]^2 dx = \pi \int_1^\infty \frac{1}{x^2} \, dx$$

$$= \pi \left[-\frac{1}{x}\right]_1^\infty$$

$$= \pi \left[\lim_{b \to \infty} -\frac{1}{b} - \left(-\frac{1}{1}\right)\right]$$

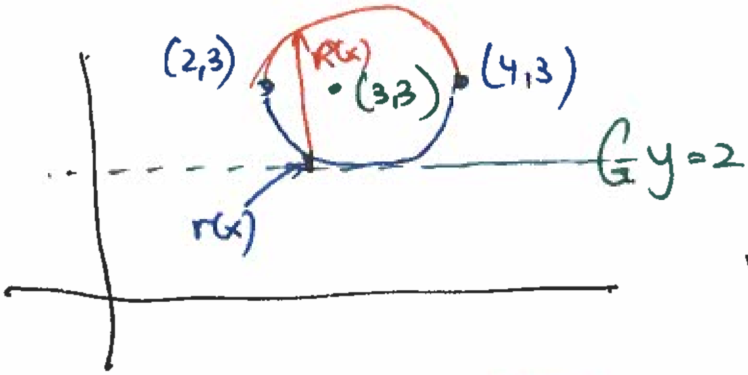
$$= \pi [0 + 1] = \boxed{\pi}$$

(4) A CIRCLE WITH $r=1$ AND CENTER $(3,3)$

HAS THE EQ'N (1) $\boxed{(x-3)^2+(y-3)^2=1}$

a) FOR THE VOLUME OF REVOLUTION ABOUT $y=2$

CONSIDER THE WASHER METHOD



THE RESULTING INTEGRAL IS

$$V=\int_2^4 \pi R^2(x)\,dx-\int_2^4 \pi r^2(x)\,dx$$

"TOP" OF CIRCLE

WHERE $R(x)=y_1(x)-2$

$r(x)=y_2(x)-2$ AXIS OF ROTATION IS $y=2$

"BOTTOM" OF CIRCLE

REARRANGING (1) FOR $y_1(x)$ + $y_2(x)$ WE GET

$$y_{1,2}(x)=3\pm\sqrt{1-(x-3)^2}$$

AND

$$R(x)=\left[3+\sqrt{1-(x-3)^2}\right]-2=1+\sqrt{1-(x-3)^2}$$

$$r(x)=\left[3-\sqrt{1-(x-3)^2}\right]-2=1-\sqrt{1-(x-3)^2}$$

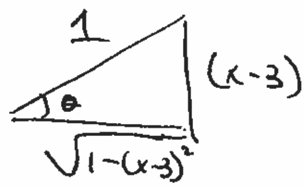
WITH THIS WE GET THAT

$$V = \pi \left[\int_2^4 \left[1 + \sqrt{1-(x-3)^2}\right]^2 dx + \int_2^4 \left[1 - \sqrt{1-(x-3)^2}\right]^2 dx \right]$$

$$= \pi \left[\int_2^4 \left[\cancel{1} + 2\sqrt{1-(x-3)^2} + \cancel{(1-(x-3)^2)}\right] dx - \int_2^4 \left(\cancel{1} - 2\sqrt{1-(x-3)^2} + \cancel{(1-(x-3)^2}\right.\right.$$

$$= \pi \left[\int_2^4 4\sqrt{1-(x-3)^2}\, dx \right] = 4\pi \int_2^4 \sqrt{1-(x-3)^2}\, dx$$

USING TRIG SUBS



(I) $\sin\theta = (x-3)$

(II) $\cos(\theta) = \sqrt{1-(x-3)^2}$

DIFF (I) WE GET:

$\cos\theta\, d\theta = dx$

SO $$V = 4\pi \int_{-\pi/2}^{\pi/2} \cos\theta \cdot \cos\theta \cdot d\theta = 4\pi \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$$

WHERE $\cos(2\theta) = \cos^2\theta - \sin^2\theta \Rightarrow \cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$V = 4\pi \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta = 2\pi \left[\theta + \frac{\sin(2\theta)}{2}\right]_{-\pi/2}^{\pi/2}$$

$$= 2\pi \left[\left(\pi/2 + \frac{0}{2}\right) - \left(-\pi/2 + \frac{0}{2}\right)\right]$$

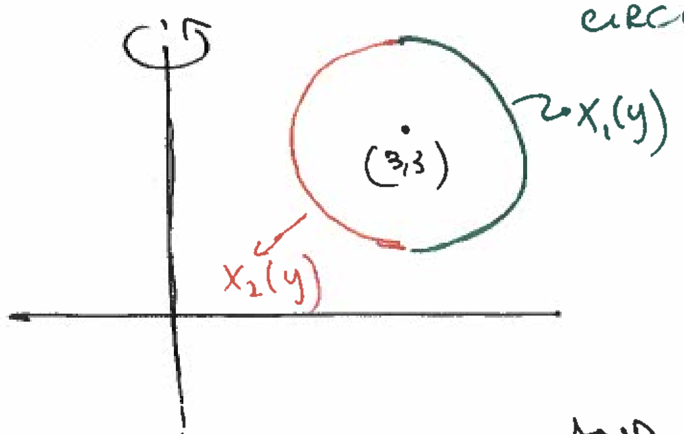
$$\boxed{V = 2\pi^2}$$

b) FOR THE SURFACE AREA OF REVOLUTION ABOUT THE LINE $x=0$

WE GET

$$S = \underbrace{2\pi \int_2^4 x_1(y) \cdot \sqrt{1+\left(\frac{dx_1}{dy}\right)^2}\, dy}_{\text{OUTER SEMI CIRCLE}} + \underbrace{2\pi \int_2^4 x_2(y) \cdot \sqrt{1+\left(\frac{dx_2}{dy}\right)^2}\, dy}_{\text{INNER SEMI CIRCLE}}$$

$x=0$



WE SOLVE (1) TO OBTAIN $x_1(y)$ & $x_2(y)$

$$x_{1,2}(y) = 3 \pm \sqrt{1-(y-3)^2}$$

AND $$\frac{dx_{1,2}}{dy} = \mp \frac{y-3}{\sqrt{1-(y-3)^2}}$$

SO THAT $$\left(\frac{dx_{1,2}}{dy}\right)^2 = \frac{(y-3)^2}{1-(y-3)^2}$$

AND $$1 + \left(\frac{dx_{1,2}}{dy}\right)^2 = \frac{1-(y-3)^2}{1-(y-3)^2} + \frac{(y-3)^2}{1-(y-3)^2} = \frac{1}{1-(y-3)^2}$$

FINALLY $$\sqrt{1+\left(\frac{dx_{1,2}}{dy}\right)^2} = \frac{1}{\sqrt{1-(y-3)^2}}$$

SUBSTITUTING $x_1(y)$, $x_2(y)$, $\frac{dx_1}{dy}$ AND $\frac{dx_2}{dy}$ WE GET

$$S = 2\pi\left[\int_2^4 \left(3+\sqrt{1-(y-3)^2}\right)\cdot\frac{1}{\sqrt{1-(y-3)^2}}\,dy\right.$$

$$\left.+\int_2^4\left(3-\sqrt{1-(y-3)^2}\right)\cdot\frac{1}{\sqrt{1-(y-3)^2}}\,dy\right]$$

$$= 2\pi\left[\int_2^4\left[\frac{3}{\sqrt{1-(y-3)^2}} + \cancel{1} + \frac{3}{\sqrt{1-(y-3)^2}} - \cancel{1}\right]dy\right]$$

$$= 12\pi\int_2^4\frac{1}{\sqrt{1-(y-3)^2}}\,dy = 12\pi\Big[\arcsin(y-3)\Big]_2^4$$

$$= 12\pi\Big[\underbrace{\arcsin(1)}_{\pi/2} - \underbrace{\arcsin(-1)}_{(-\pi/2)}\Big]$$

$$= 12\pi\cdot[\pi] = \boxed{12\pi^2}$$

5a)

$$\int_{-3}^{2} \frac{1}{w^4}\,dw$$

NOTE: $f(w)=\frac{1}{w^4}$ IS UNDEFINED @ $w=0$ AND THE INTERVAL OF INTEGRATION IS FROM $a=-3$ TO $b=$

$$= \int_{-3}^{0} \frac{1}{w^4}\,dw + \int_{0}^{2} \frac{1}{w^4}\,dw$$

$$= \lim_{b\to 0^-} \int_{-3}^{b} \frac{1}{w^4}\,dw + \lim_{a\to 0^+} \int_{a}^{2} \frac{1}{w^4}\,dw$$

BUT $\int_{a}^{0} \frac{1}{x^p}\,dx$ AND $\int_{0}^{b} \frac{1}{x^p}\,dx$ DIVERGE IF $p\geq 1$

SO $\int_{-3}^{0} \frac{1}{w^4}\,dw$ AND $\int_{0}^{2} 1/w^4\,dw$ DIVERGE

SO $\int_{-3}^{2} \frac{1}{w^4}\,dw$ MUST DIVERGE

5b)

$$\int_{-\infty}^{\infty} y e^{-y^2}\,dy = -\frac{1}{2}e^{-y^2}\Big|_{-\infty}^{\infty}$$

$$= \lim_{a\to\infty}\left[-\frac{1}{2}e^{-a^2} + \frac{1}{2}e^{-a^2}\right] = 0$$

SO THE INTEGRAL CONVERGES

[USING GEOMETRY: $f(y)=ye^{-y^2}$ IS AN ODD FUNCTION INTEGRATED OVER A SYMMETRIC INTERVAL

SO $\int_{-a}^{a} f(y)\,dy = 0$]

5d|

$$\int_1^\infty \frac{\log(x)}{x}\,dx = \left(\log\left[\log(x)\right] + K\right)\Big|_1^\infty$$

$$= \lim_{b\to\infty}\left[\log(\log(b)) \cancel{+K}\right] - \left[\log(\log(1)) \cancel{+K}\right]$$

As $b\to\infty$, $\log(b)\to\infty$ + $\log(1) = 0$

so $\log(\log(b))\to\infty$ $\log(\log(1)) \to -\infty$

so $\int_1^\infty \frac{\log(x)}{x}\,dx$ MUST DIVERGE

Note: INTERVAL OF INTEGRATION IS $a=-1$ AND $b=14$ SO FINITE. AND $f(z) = (z+2)^{-1/4}$ IS DEFINED IN THAT INTERVAL

5e||

$$\int_{-1}^{14} \frac{1}{\sqrt[4]{z+2}}\,dz = \int_{-1}^{14} (z+2)^{-1/4}\,dz$$

$$= \left(\frac{4}{3}(z+2)^{3/4} + K\right)\Big|_{-1}^{14}$$

$$= \left(\frac{4}{3}(14+2)^{3/4} \cancel{+K}\right) - \left(\frac{4}{3}(-1+2)^{3/4} \cancel{+K}\right)$$

$$= \frac{4}{3}(16)^{3/4} - \frac{4}{3}(1)^{3/4} = \frac{4}{3}(8) - \frac{4}{3}(1) = \frac{28}{3}$$

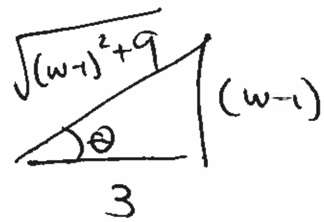
so $\int_{-1}^{14} \frac{1}{\sqrt[4]{z+2}}\,dz$ CONVERGES

5f)

For $\displaystyle\int_1^{\infty} \frac{1}{w^2-2w+10}\,dw = \int_1^{\infty} \frac{1}{(w^2-2w+1)+9}\,dw$

$$= \int_1^{\infty} \frac{1}{(w-1)^2+9}\,dw$$

Using Δ's



① $\cos(\theta) = \dfrac{3}{\sqrt{(w-1)^2+9}}$

② $\sin(\theta) = \dfrac{w-1}{\sqrt{(w-1)^2+9}}$

③ $\tan\theta = \dfrac{w-1}{3}$

From ① $\quad \dfrac{\cos(\theta)}{3} = \dfrac{1}{\sqrt{(w-1)^2+9}} \rightarrow \dfrac{\cos^2\theta}{9} = \dfrac{1}{(w-1)^2+9}$

Diff ③ $\quad \dfrac{1}{\cos^2\theta}\,d\theta = \dfrac{1}{3}\,dw \rightarrow \dfrac{3}{\cos^2\theta}\,d\theta = dw$

If $w_1 = 1 \rightarrow \tan\theta_1 = \dfrac{1-1}{3} \rightarrow \theta_1 = 0$

$w_2 \rightarrow \infty \rightarrow \tan\theta_2 \rightarrow \infty \rightarrow \theta_2 = \pi/2$

$$\int_1^{\infty} \frac{1}{(w-1)^2+9}\,dw = \int_0^{\pi/2} \frac{\cos^2\theta}{9} \cdot \frac{3}{\cos^2\theta}\,d\theta = \frac{1}{3}\int_0^{\pi/2} d\theta$$

$$= \frac{1}{3}\left[\theta + k\right]_0^{\pi/2} = \frac{1}{3}\left(\pi/2 \cancel{+k}\right) - 1/3\left(0 \cancel{+k}\right)$$

$$= \pi/6$$

$\boxed{\text{Converges}}$

5h) $\displaystyle\int_0^1 \frac{x}{(x+1)^4}\,dx$

Note: Interval of integration $0 \le x \le 1$ [finite]

$f(x) = \dfrac{x}{(x+1)^4}$ is defined in $0 \le x \le 1$

↳ Integral must converge

Let $u = x+1 \rightarrow u - 1 = x$

$du = dx$

$x_1 = 0 \rightarrow u_1 = 0+1 = 1$

$x_2 = 1 \rightarrow u_2 = 1+1 = 2$

So

$$\int_0^1 \frac{x}{(x+1)^4}\,dx = \int_1^2 \frac{u-1}{u^4}\,du = \int_1^2 \frac{1}{u^3}\,du - \int_1^2 \frac{1}{u^4}\,du$$

$$= \left[-\frac{1}{2u^2} \cancel{+K}\right]_1^2 - \left[-\frac{1}{3u^3} \cancel{+K}\right]_1^2$$

$$= \left(-\frac{1}{2(2)^2} \cancel{+K}\right) - \left(-\frac{1}{2(1)^2} \cancel{+K}\right) - \left[\left(-\frac{1}{3(2)^3} \cancel{+K}\right) - \left(-\frac{1}{3(1)^3} \cancel{+K}\right)\right]$$

$$= \left(-\frac{1}{8} + \frac{1}{2}\right) - \left(-\frac{1}{24} + \frac{1}{3}\right)$$

$$= \frac{3}{8} - \frac{7}{24} = \frac{2}{24} = \boxed{\frac{1}{12}}$$

5 i ll

$$\int_{-3}^{3} \frac{1}{x^2+x}\,dx$$

NOTE: ALTHOUGH INTERVAL OF INTEGRATION IS FINITE $[-3 \le x \le 3]$

$f(x) = \frac{1}{x^2+x} = \frac{1}{x(x+1)}$ IS UNDEFINED @ $x=0$ + $x=1$

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$ [USING PARTIAL FRACTIONS]

WHERE $1 = A(x+1) + Bx$

SO $A=1$ + $B=-1$

AND $$\int_{-3}^{3} \frac{1}{x^2+x}\,dx = \int_{-3}^{3} \frac{1}{x}\,dx - \int_{-3}^{3} \frac{1}{x+1}\,dx$$

LET $u=x+1$
$du=dx$

$$= \underbrace{\int_{-3}^{3} \frac{1}{x}\,dx}_{A} - \underbrace{\int_{2}^{4} \frac{1}{u}\,du}_{B}$$

BOTH A AND B ARE INTEGRALS OF THE TYPE

$\int_a^b x^{-p}\,dx$ WITH $p=1$ WHERE THE INTERVAL INCLUDES $x=0$ $\Rightarrow$ DIVERGENT INTEGRALS

6 a) $\dfrac{dz}{du} - \dfrac{e^{u-z}}{1+e^u} = 0$ WHERE $z(1) = 0$

SO $\dfrac{dz}{du} = \dfrac{1}{e^z} \cdot \dfrac{e^u}{1+e^u}$

SEPARATE VBLS

$$e^z dz = \frac{e^u}{1+e^u} du$$

INTEGRATE

$$\int e^z dz = \int \frac{e^u}{1+e^u} du$$

$$e^z = \log[1+e^u] + K$$

SINCE $z(1) = 0 \rightarrow$ $u=1$, $z=0$

$$e^0 = \log[1+e^1] + K \rightarrow 1 - \log(1+e) = K$$

SO $e^z = \log[1+e^u] + 1 - \log(1+e)$

OR

$$\boxed{z(u) = \log\left[\log(1+e^u) + 1 - \log(1+e)\right]}$$

6c ||

FOR $\frac{dT}{dx} = \sqrt{Tx}$ WHERE $T(1) = 2$

$$\frac{dT}{dx} = \sqrt{T} \cdot \sqrt{x}$$

SEPARATE VBLS

$$\frac{dT}{\sqrt{T}} = \sqrt{x}\,dx$$

$$\frac{1}{T^{1/2}}\,dT = x^{1/2}\,dx$$

INTEGRATE BOTH SIDES

$$2T^{1/2} = \frac{2}{3}x^{3/2} + K$$

SO IF $T(1) = 2 \Rightarrow \begin{matrix} x=1 \\ T=2 \end{matrix}$

$$2\sqrt{2} = \frac{2}{3}(1)^{3/2} + K \Rightarrow K = 2\sqrt{2} - \frac{2}{3} = \frac{6\sqrt{2}-2}{3}$$

SO $2T^{1/2} = \frac{2}{3}x^{3/2} + \frac{6\sqrt{2}-2}{3}$

OR $T^{1/2} = \frac{1}{3}x^{3/2} + \frac{3\sqrt{2}-1}{3}$

OR $\boxed{T(x) = \left[\frac{x^{3/2} + 3\sqrt{2} - 1}{3}\right]^2}$

6e11.

FOR $\frac{dy}{dx} = \frac{e^x}{(1+y)^2}$ WHERE $y(0)=0$

SEPARATE VBLS

$$(1+y)^2 dy = e^x dx$$

INTEGRATE BOTH SIDES

$$\int (1+y)^2 dy = \int e^x dx$$

$$\frac{1}{3}(1+y)^3 = e^x + K$$

SO FOR $y(0)=0 \Rightarrow \begin{matrix} x=0 \\ y=0 \end{matrix}$

$$\frac{1}{3}(1+0)^3 = e^0 + K \Rightarrow K = \frac{1}{3} - 1 = -\frac{2}{3}$$

SO $\frac{1}{3}(1+y)^3 = e^x - \frac{2}{3}$

OR $(1+y)^3 = 3e^x - 2$

$$1+y = \sqrt[3]{3e^x - 2}$$

OR

$$y(x) = \sqrt[3]{3e^x - 2} - 1$$