Calculus

Calculus - the study of change, as related to functions

Formally co-developed around the 1660's by Newton and Leibniz

Two main branches - differential and integral

Central role in much of modern science

Physics, especially kinematics and electrodynamics

Economics, engineering, medicine, chemistry, *etc.*

Functions - Review

If calculus is a new language, functions are its words!

Must be ***'fluent'*** in elementary functions

1) polynomials

Functions - Review

If calculus is a new language, functions are its words!

Must be **'*fluent*'** in elementary functions

2) trigonometric

Functions - Review

If calculus is a new language, functions are its words!

Must be '***fluent***' in elementary functions

3) exponential

Functions - Review

If calculus is a new language, functions are its words!

Must be ‘***fluent***’ in elementary functions

4) logarithmic

Functions - Review

If calculus is a new language, functions are its words!

Must be ‘***fluent***’ in elementary functions

5) fractional

Integration Techniques

Main strategy for integration

(*) Know your derivatives forward and backward! (*)

Unfortunately, this severly limits the number of integrals we can do

If (*) ______ (*) fails, our options are limited!

(1): Look clever substitutions to get back to (*) ______ (*)

(2): Look for special forms to get back to (*) ______ (*)

(3): Exploit the symmetry of the definite integral

Integration - Derivative Substitutions

Consider the following integrals

$$I_1 = \int \frac{1}{1+x} dx \qquad\qquad I_2 = \int \frac{1}{1+x^2} dx$$

Integration - Derivative Substitutions

Recall the chain rule

$$\frac{d}{dx}\left[F(g(x))\right] = \left.\frac{dF}{dx}\right|_{x=g(x)} g'(x) = f(g(x))g'(x)$$

Integrating the LHS and the RHS we get

$$F(g(x)) = \int f[g(x)]g'(x)dx$$

Suggests a method of solving integrals of this form, provided we have a suitable antiderivative

In general, we 'transform' the integral via a clever substitution

If $\boxed{① \; u = g(x)} \Rightarrow \boxed{② \quad \frac{du}{dx} = g'(x)}$ then

$$\int f[g(x)]g'(x)dx = \int f(u)\frac{du}{dx}dx = \int f(u)du$$

If the antiderivative of f is known, the problem is (essentially) solved

Derivative Substitutions - Examples

Example 1: Calculate

$$\int \frac{x}{1+x^2}dx$$

Derivative Substitutions - Examples

Example 2: Calculate

$$\int \tan\theta \, d\theta$$

Derivative Substitutions - Examples

Example 3: Calculate

$$\int t^2 \csc(t^3) \cot(t^3) \, dt$$

Derivative Substitutions - Examples

Example 4: Calculate

$$\int e^{x/5}\, dx$$

Derivative Substitutions - Definite Integrals

If we are facing a definite integral of the form

$$I = \int_a^b f[g(x)]g'(x)dx$$

the previous procedure requires an extra step to transform the limits

If ① $\boxed{u = g(x)}$ $\Rightarrow$ ② $\boxed{\frac{du}{dx} = g'(x)}$ and ③ $\boxed{u_1 = g(a),\ u_2 = g(b)}$ then

$$\int_a^b f[g(x)]g'(x)dx = \int_a^b f(u)\frac{du}{dx}dx = \int_{u_1}^{u_2} f(u)du = F(u_2) - F(u_1) = I$$

Derivative Substitutions - Examples

Example 5: Evaluate

$$\int_1^{e^2} \frac{\ln x}{x} \, dx$$

Integration via Parts

We have seen how to integrate when

① the integrand has a known antiderivative

② the integrand contains a function and its derivative

What about

$$\int x \sin x \; dx \qquad \text{or} \qquad \int e^t \sin t \; dt$$

Integration by Parts - Motivation

Last time, we looked at the chain rule for inspiration.

This time, lets look at the product rule!

$$\frac{d}{dx}[fg] = \frac{df}{dx}g + f\frac{dg}{dx}$$

$$\int f\frac{dg}{dx}dx = fg - \int g\frac{df}{dx}dx$$

Integration by Parts - Examples

Example 1: Find

$$\int x \sin x \ dx$$

Integration by Parts - Examples

Example 2: Find

$$\int x^3 \sin x \; dx$$

Integration by Parts - Examples

Example 3: Find

$$\int x^2 e^{4x^2}\ dx$$

Integration by Parts - Examples

Example 4: Find

$$\int \log t \; dt$$

Integration by Parts - Examples

Example 5: Evaluate

$$\int_0^1 \frac{x^2+1}{e^x}\ dt$$

Integration by Parts - Examples

Example 6: Find

$$\int e^t \sin 2t \; dt$$

Integration of Trigonometric Functions

Our integration repertoire now includes integrals where

① the integrand has a known antiderivative

② the integrand contains a function and its derivative

③ the integrand is a 'special' product of functions

$$\int \sin\theta \cos\theta \; d\theta$$

Trigonometric Functions - Using Identities

Often times, we can use trigonometric identities to reformulate an integral

$$\int \sin\theta \cos\theta \; d\theta$$

Trigonometric Functions - Using Identities

Odd powers of trigonometric functions

① $\int \cos^3 t \; dt$

② $\int \cos^5 t \; dt$

Trigonometric Functions - Using Identities

What about even powers?

③ $\int \cos^2 t \; dt$

Trigonometric Functions - Using Identities

What about even powers?

④ $\int \cos^4 t \; dt$

Trigonometric Functions - Using Identities

Similar strategies apply for integrals involving $\tan\theta$ and $\sec\theta$

$$\text{Use } 1+\tan^2\theta = \sec^2\theta,\ \frac{d}{d\theta}\tan\theta = \sec^2\theta \text{ and } \frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta$$

④ $\displaystyle\int \tan^5 t \sec^3 t \; dt$

Trigonometric Functions - Examples

Example 1: Find

$$\int \cos^2 x \sin^2 x \; dx$$

Trigonometric Functions - Examples

Example 2: Find

$$\int \tan^2 \theta \ d\theta$$

Trigonometric Functions - Examples

Example 3: Find

$$\int \frac{1}{1 - \sin\theta} \, d\theta$$

Trigonometric Functions - Examples

Example 4: Find

$$\int_0^{\pi/4} \sqrt{1 - \cos 4\theta}\; d\theta$$

Integration via Partial Fractions

When an integrand of the form $\dfrac{P(x)}{Q(x)}$ with polynomials $P(x)$ and $Q(x)$

① if $P(x) = cQ'(x)$ then we can use a derivative substitution

② else, we have to be more clever!

Partial Fractions - Aside

Consider the following integral

$$\int \left[\frac{1}{x-1} - \frac{1}{x}\right] \, dx$$

Partial Fractions - Procedure - $n < m$

If we consider

$$\int \frac{P(x)}{Q(x)}\,dx$$

with $\deg(P) = n$, $\deg(Q) = m$ and $n < m$

① Factor $Q(x) = q_1(x)q_2(x)\dots q_m(x)$ with $\deg(q_i) = d_i$

② Write $\frac{P(x)}{Q(x)} = \frac{p_1(x)}{q_1(x)} + \frac{p_2(x)}{q_2(x)} + \dots + \frac{p_m(x)}{q_m(x)}$ with $\deg(p_i) < d_i$

③ Using the equality in ②, solve for each of the $p_i(x)$

④ Integrate each of the individual partial fractions

Partial Fractions

Denominator - $q_i(x)$	Partial Fraction - $\frac{p_i(x)}{q_i(x)}$
$(x-a)$	$\frac{A}{x-a}$
$(x-a)^r$	$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_r}{(x-a)^r}$
(ax^2+bx+c)	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^r$	$\frac{A_1x+B_1}{(ax^2+bx+c)} + \ldots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$

Partial Fractions - Examples

Example 1: Find

$$\int \frac{y}{2y^2 - 3y + 1}\, dy$$

Partial Fractions - Examples

Example 2: Find

$$\int \frac{10}{5x^2 - 2x^3}\ dx$$

Partial Fractions - Examples

Example 3: Find

$$\int \frac{10}{(x-1)(x^2+9)} \, dx$$

Partial Fractions - Procedure - $n \geq m$

If we consider

$$\int \frac{P(x)}{Q(x)}\, dx$$

with $\deg(P) = n$, $\deg(Q) = m$ and $n \geq m$ we add an extra step!

⓪ Long division $\Rightarrow P(x) = g(x) + \frac{R(x)}{Q(x)}$ with $\deg(R) < \deg(Q)$

① Factor $Q(x) = q_1(x)q_2(x)\ldots q_m(x)$ with $\deg(q_i) = d_i$

② Write $\frac{R(x)}{Q(x)} = \frac{r_1(x)}{q_1(x)} + \frac{r_2(x)}{q_2(x)} + \ldots + \frac{r_m(x)}{q_m(x)}$ with $\deg(r_i) < d_i$

③ Using the equality in ②, solve for each of the $r_i(x)$

④ Integrate $g(x)$ and each of the individual partial fractions

Partial Fractions - Examples

Example 4: Find

$$\int_3^4 \frac{x^3 - 2x^2 - 4}{x^3 - 2x^2}\ dx$$

Partial Fractions - Examples

Example 5: Find

$$\int \frac{x^5 + x - 1}{x^3 + 1}\, dx$$

Partial Fractions - Examples

Example 6: Find

$$\int \frac{1}{1+e^x}\,dx$$

Integration via Trigonometric Substitution

We have strategies for integrals involving

① known antiderivatives and ② derivative substitutions

Unfortunately, unlike differentiation, integration is '*unstable*'

Small changes to the integrand can lead to difficult integrals

Need to develop new techniques!

Trigonometric Substitutions - Methods

Consider integrals that contain the following forms and their powers

$$\sqrt{a^2 - x^2}, \quad \sqrt{x^2 + a^2} \quad \text{and} \quad \sqrt{x^2 - a^2}, \text{ with } a > 0$$

We look for substitutions that simplify such integrals

① Right triangles to extract the substitution

② Trigonometric identities

Trigonometric Substitutions - Triangles

The 'Pythegorean' form of the integrand suggests the related right triangle

Integrand	Triangle
$\sqrt{x^2+a^2}$, $\frac{1}{\sqrt{x^2+a^2}}$, $\frac{1}{x^2+a^2}$	
$\sqrt{a^2-x^2}$, $\frac{1}{\sqrt{a^2-x^2}}$	
$\sqrt{x^2-a^2}$, $\frac{1}{\sqrt{x^2-a^2}}$	

Trigonometric Substitutions - Identities

Alternatively, when the integrand contains

$$(\sqrt{a^2 - x^2})^{\pm n}, \;(\sqrt{x^2 - a^2})^{\pm n}, \text{ or } (\sqrt{x^2 + a^2})^{\pm n}, \; a > 0$$

we can use the various forms of the 'Pythegorean' trig identity to find a suitable substitution

① $\cos^2\theta + \sin^2\theta = 1$

② $1 + \tan^2\theta = \sec^2\theta$

Trigonometric Substitutions - Identities

This suggests the following set of substitutions

Integrand	Substitution
$\sqrt{a^2 - x^2}$, $\frac{1}{\sqrt{a^2-x^2}}$	$x = a\sin\theta$
$\sqrt{x^2 + a^2}$, $\frac{1}{\sqrt{x^2+a^2}}$, $\frac{1}{x^2+a^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$, $\frac{1}{\sqrt{x^2-a^2}}$	$x = a\sec\theta$

Trigonometric Substitutions - Examples

Example 1: Find

$$\int \frac{1}{\sqrt{9-x^2}}\,dx$$

Trigonometric Substitutions - Examples

Example 2: Find

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$$

Trigonometric Substitutions - Examples

Example 3: Find

$$\int \frac{1}{\sqrt{x^2 - 6x}}\, dx$$

Trigonometric Substitutions - Examples

Example 4: Evaluate

$$\int_0^{\frac{\sqrt{2}}{2}} \frac{x^2}{(1-x^2)^{3/2}}\, dx$$