Calculus - the study of change, as related to functions
Formally co-developed around the 1660’s by Newton and Leibniz
Two main branches - differential and integral

Central role in much of modern science
Physics, especially kinematics and electrodynamics
Economics, engineering, medicine, chemistry, etc.
Functions - Review

If calculus is a new language, functions are its words! Must be ‘fluent’ in elementary functions

1) polynomials
If calculus is a new language, functions are its words!
   Must be ‘fluent’ in elementary functions

2) trigonometric
Functions - Review

If calculus is a new language, functions are its words!
   Must be ‘fluent’ in elementary functions

3) exponential
If calculus is a new language, functions are its words!
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4) logarithmic
If calculus is a new language, functions are its words!
  Must be ‘fluent’ in elementary functions
5) fractional
Integration Techniques

Main strategy for integration

- Know your derivatives forward and backward!

Unfortunately, this severely limits the number of integrals we can do

If fails, our options are limited!

1: Look clever substitutions to get back to

2: Look for special forms to get back to

3: Exploit the symmetry of the definite integral
Integration - Derivative Substitutions

Consider the following integrals

\[ I_1 = \int \frac{1}{1 + x} \, dx \]

\[ I_2 = \int \frac{1}{1 + x^2} \, dx \]
Integration - Derivative Substitutions

Recall the chain rule
\[ \frac{d}{dx} [F(g(x))] = \left. \frac{dF}{dx} \right|_{x=g(x)} g'(x) = f(g(x))g'(x) \]

Integrating the LHS and the RHS we get
\[ F(g(x)) = \int f[g(x)]g'(x) \, dx \]

Suggests a method of solving integrals of this form, provided we have a suitable antiderivative

In general, we ‘transform’ the integral via a clever substitution

If \( u = g(x) \) \( \Rightarrow \)

\( \frac{du}{dx} = g'(x) \) then

\[ \int f[g(x)]g'(x) \, dx = \int f(u) \frac{du}{dx} \, dx = \int f(u) \, du \]

If the antiderivative of \( f \) is known, the problem is (essentially) solved
Derivative Substitutions - Examples

Example 1: Calculate

$$\int \frac{x}{1 + x^2} \, dx$$
Example 2: Calculate

$$\int \tan \theta \, d\theta$$
Example 3: Calculate

\[ \int t^2 \csc(t^3) \cot(t^3) \, dt \]
Example 4: Calculate

\[ \int e^{x/5} \, dx \]
Derivative Substitutions - Definite Integrals

If we are facing a definite integral of the form

$$I = \int_{a}^{b} f[g(x)]g'(x)\,dx$$

the previous procedure requires an extra step to transform the limits

If \(1\) \(u = g(x)\) \(\Rightarrow\) \(2\) \(\frac{du}{dx} = g'(x)\) and \(3\) \(u_1 = g(a), u_2 = g(b)\) then

$$\int_{a}^{b} f[g(x)]g'(x)\,dx = \int_{a}^{b} f(u)\frac{du}{dx}\,dx = \int_{u_1}^{u_2} f(u)\,du = F(u_2) - F(u_1) = I$$
Example 5: Evaluate

$$\int_{1}^{e^2} \frac{\ln x}{x} \, dx$$
Integration via Parts

We have seen how to integrate when

1. the integrand has a known antiderivative

What about

\[ \int x \sin x \, dx \quad \text{or} \quad \int e^t \sin t \, dt \]
Integration by Parts - Motivation

Last time, we looked at the chain rule for inspiration. This time, let's look at the product rule!

\[
\frac{d}{dx}[fg] = \frac{df}{dx}g + f\frac{dg}{dx}
\]

\[
\int f\frac{dg}{dx}dx = fg - \int g\frac{df}{dx}dx
\]
Integration by Parts - Examples

Example 1: Find

\[ \int x \sin x \, dx \]
Example 2: Find

\[ \int x^3 \sin x \, dx \]
Example 3: Find

\[ \int x^2 e^{4x^2} \, dx \]
Example 4: Find

$$\int \log t \, dt$$
Example 5: Evaluate

\[ \int_{0}^{1} \frac{x^2 + 1}{e^x} \, dt \]
Integration by Parts - Examples

Example 6: Find

\[ \int e^t \sin 2t \, dt \]
Integration of Trigonometric Functions

Our integration repertoire now includes integrals where

① the integrand has a known antiderivative

② the integrand contains a function and its derivative

③ the integrand is a ‘special’ product of functions

\[ \int \sin \theta \cos \theta \ d\theta \]
Trigonometric Functions - Using Identities

Often times, we can use trigonometric identities to reformulate an integral

\[ \int \sin \theta \cos \theta \, d\theta \]
Trigonometric Functions - Using Identities

Odd powers of trigonometric functions

① \[ \int \cos^3 t \, dt \]

② \[ \int \cos^5 t \, dt \]
Trigonometric Functions - Using Identities

What about even powers?

\( \int \cos^2 t \, dt \)
What about even powers?

\[ \int \cos^4 t \ dt \]
Trigonometric Functions - Using Identities

Similar strategies apply for integrals involving $\tan \theta$ and $\sec \theta$

Use $1 + \tan^2 \theta = \sec^2 \theta$, $\frac{d}{d\theta} \tan \theta = \sec^2 \theta$ and $\frac{d}{d\theta} \sec \theta = \sec \theta \tan \theta$

$\int \tan^5 t \sec^3 t \, dt$
Example 1: Find

\[ \int \cos^2 x \sin^2 x \, dx \]
Example 2: Find

\[ \int \tan^2 \theta \, d\theta \]
Example 3: Find

\[ \int \frac{1}{1 - \sin \theta} \, d\theta \]
Trigonometric Functions - Examples

Example 4: Find

\[ \int_{0}^{\pi/4} \sqrt{1 - \cos 4\theta} \, d\theta \]
Integration via Partial Fractions

When an integrand of the form $\frac{P(x)}{Q(x)}$ with polynomials $P(x)$ and $Q(x)$

① if $P(x) = cQ'(x)$ then we can use a derivative substitution

② else, we have to be more clever!
Partial Fractions - Aside

Consider the following integral

\[ \int \left[ \frac{1}{x-1} - \frac{1}{x} \right] \, dx \]
Partial Fractions - Procedure - $n < m$

If we consider

$$\int \frac{P(x)}{Q(x)} \, dx$$

with $\deg(P) = n$, $\deg(Q) = m$ and $n < m$

① Factor $Q(x) = q_1(x)q_2(x) \ldots q_m(x)$ with $\deg(q_i) = d_i$

② Write $\frac{P(x)}{Q(x)} = \frac{p_1(x)}{q_1(x)} + \frac{p_2(x)}{q_2(x)} + \ldots + \frac{p_m(x)}{q_m(x)}$ with $\deg(p_i) < d_i$

③ Using the equality in ②, solve for each of the $p_i(x)$

④ Integrate each of the individual partial fractions
### Partial Fractions

<table>
<thead>
<tr>
<th>Denominator - $q_i(x)$</th>
<th>Partial Fraction - $\frac{p_i(x)}{q_i(x)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x - a)$</td>
<td>$\frac{A}{x-a}$</td>
</tr>
<tr>
<td>$(x - a)^r$</td>
<td>$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_r}{(x-a)^r}$</td>
</tr>
<tr>
<td>$(ax^2 + bx + c)$</td>
<td>$\frac{Ax+B}{ax^2+bx+c}$</td>
</tr>
<tr>
<td>$(ax^2 + bx + c)^r$</td>
<td>$\frac{A_1x+B_1}{(ax^2+bx+c)} + \ldots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$</td>
</tr>
</tbody>
</table>
Partial Fractions - Examples

Example 1: Find

\[
\int \frac{y}{2y^2 - 3y + 1} \, dy
\]
Partial Fractions - Examples

Example 2: Find

$$\int \frac{10}{5x^2 - 2x^3} \, dx$$
Partial Fractions - Examples

Example 3: Find

\[ \int \frac{10}{(x - 1)(x^2 + 9)} \, dx \]
Partial Fractions - Procedure - \( n \geq m \)

If we consider

\[ \int \frac{P(x)}{Q(x)} \, dx \]

with \( \deg(P) = n \), \( \deg(Q) = m \) and \( n \geq m \) we add an extra step!

0. Long division \( \Rightarrow P(x) = g(x) + \frac{R(x)}{Q(x)} \) with \( \deg(R) < \deg(Q) \)

1. Factor \( Q(x) = q_1(x)q_2(x) \ldots q_m(x) \) with \( \deg(q_i) = d_i \)

2. Write \( \frac{R(x)}{Q(x)} = \frac{r_1(x)}{q_1(x)} + \frac{r_2(x)}{q_2(x)} + \ldots + \frac{r_m(x)}{q_m(x)} \) with \( \deg(r_i) < d_i \)

3. Using the equality in 2, solve for each of the \( r_i(x) \)

4. Integrate \( g(x) \) and each of the individual partial fractions
Partial Fractions - Examples

Example 4: Find

\[ \int_3^4 \frac{x^3 - 2x^2 - 4}{x^3 - 2x^2} \, dx \]
Partial Fractions - Examples

Example 5: Find

$$\int \frac{x^5 + x - 1}{x^3 + 1} \, dx$$
Partial Fractions - Examples

Example 6: Find

\[ \int \frac{1}{1 + e^x} \, dx \]
Integration via Trigonometric Substitution

We have strategies for integrals involving

① known antiderivatives and ② derivative substitutions

Unfortunately, unlike differentiation, integration is ‘unstable’

Small changes to the integrand can lead to difficult integrals

Need to develop new techniques!
Trigonometric Substitutions - Methods

Consider integrals that contain the following forms and their powers

\[ \sqrt{a^2 - x^2}, \quad \sqrt{x^2 + a^2} \quad \text{and} \quad \sqrt{x^2 - a^2}, \text{ with } a > 0 \]

We look for substitutions that simplify such integrals

① Right triangles to extract the substitution

② Trigonometric identities
Trigonometric Substitutions - Triangles

The ‘Pythagorean’ form of the integrand suggests the related right triangle

<table>
<thead>
<tr>
<th>Integrand</th>
<th>Triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{x^2 + a^2}, \frac{1}{\sqrt{x^2 + a^2}}, \frac{1}{x^2 + a^2} )</td>
<td></td>
</tr>
<tr>
<td>( \sqrt{a^2 - x^2}, \frac{1}{\sqrt{a^2 - x^2}} )</td>
<td></td>
</tr>
<tr>
<td>( \sqrt{x^2 - a^2}, \frac{1}{\sqrt{x^2 - a^2}} )</td>
<td></td>
</tr>
</tbody>
</table>
Trigonometric Substitutions - Identities

Alternatively, when the integrand contains

\[(\sqrt{a^2 - x^2})^{\pm n}, (\sqrt{x^2 - a^2})^{\pm n}, \text{ or } (\sqrt{x^2 + a^2})^{\pm n}, a > 0\]

we can use the various forms of the ‘Pythagorean’ trig identity to find a suitable substitution

\[\cos^2 \theta + \sin^2 \theta = 1\]

\[1 + \tan^2 \theta = \sec^2 \theta\]
This suggests the following set of substitutions

<table>
<thead>
<tr>
<th>Integrand</th>
<th>Substitution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{a^2 - x^2}$, $\frac{1}{\sqrt{a^2 - x^2}}$</td>
<td>$x = a \sin \theta$</td>
</tr>
<tr>
<td>$\sqrt{x^2 + a^2}$, $\frac{1}{\sqrt{x^2 + a^2}}$, $\frac{1}{x^2 + a^2}$</td>
<td>$x = a \tan \theta$</td>
</tr>
<tr>
<td>$\sqrt{x^2 - a^2}$, $\frac{1}{\sqrt{x^2 - a^2}}$</td>
<td>$x = a \sec \theta$</td>
</tr>
</tbody>
</table>
Example 1: Find

\[ \int \frac{1}{\sqrt{9 - x^2}} \, dx \]
Trigonometric Substitutions - Examples

Example 2: Find

\[ \int \frac{1}{x^2 \sqrt{x^2 + 4}} \, dx \]
Example 3: Find

$$\int \frac{1}{\sqrt{x^2 - 6x}} \, dx$$
Trigonometric Substitutions - Examples

Example 4: Evaluate

\[\int_0^{\frac{\sqrt{2}}{2}} \frac{x^2}{(1 - x^2)^{3/2}} \, dx\]