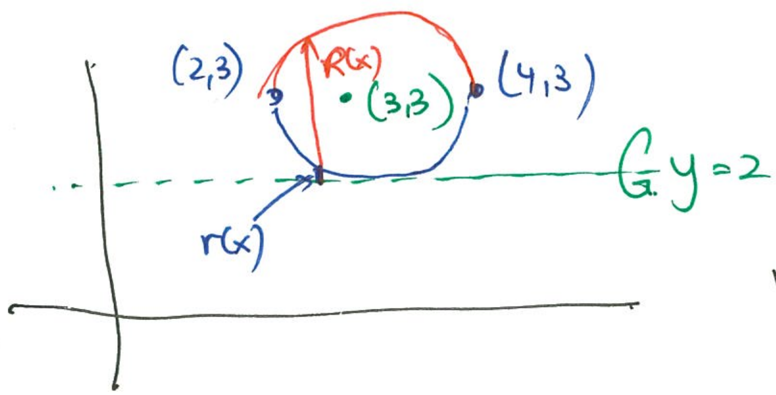
(4) A CIRCLE WITH $r = 1$ AND CENTER $(3,3)$

HAS THE EQ'N (1) $\boxed{(x-3)^2 + (y-3)^2 = 1}$

a) FOR THE VOLUME OF REVOLUTION ABOUT $y = 2$

CONSIDER THE WASHER METHOD



THE RESULTING INTEGRAL IS

$$V = \int_2^4 \pi R^2(x)\,dx - \int_2^4 \pi r^2(x)\,dx$$

"TOP" OF CIRCLE

WHERE $R(x) = y_1(x) - 2$

$r(x) = y_2(x) - 2$ AXIS OF ROTATION IS $y = 2$

"BOTTOM" OF CIRCLE

REARRANGING (1) FOR $y_1(x)$ + $y_2(x)$ WE GET

$$y_{1,2}(x) = 3 \pm \sqrt{1-(x-3)^2}$$

AND

$$R_*(x) = \left[3 + \sqrt{1-(x-3)^2}\right] - 2 = 1 + \sqrt{1-(x-3)^2}$$

$$r(x) = \left[3 - \sqrt{1-(x-3)^2}\right] - 2 = 1 - \sqrt{1-(x-3)^2}$$

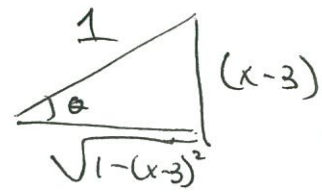
WITH THIS WE GET THAT

$$V = \pi \left[\int_2^4 \left[1 + \sqrt{1-(x-3)^2} \right]^2 dx + \int_2^4 \left[1 - \sqrt{1-(x-3)^2} \right]^2 dx \right]$$

$$= \pi \left[\int_2^4 \left[\cancel{1} + 2\sqrt{1-(x-3)^2} + \cancel{\left(1-(x-3)^2\right)} \right] dx - \int_2^4 \left(\cancel{1} - 2\sqrt{1-(x-3)^2} + \cancel{\left(1-(x-3\right.} \right.$$

$$= \pi \left[\int_2^4 4\sqrt{1-(x-3)^2}\, dx \right] = 4\pi \int_2^4 \sqrt{1-(x-3)^2}\, dx$$

USING TRIG SUBS



(i) $\sin\theta = (x-3)$

(ii) $\cos(\theta) = \sqrt{1-(x-3)^2}$

DIFF (i) WE GET:

$\cos\theta\, d\theta = dx$

SO $$V = 4\pi \int_{-\pi/2}^{\pi/2} \cos\theta \cdot \cos\theta \cdot d\theta = 4\pi \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$$

WHERE $\cos(2\theta) = \cos^2\theta - \sin^2\theta \Rightarrow \cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$V = 4\pi \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta = 2\pi \left[\theta + \frac{\sin(2\theta)}{2} \right]_{-\pi/2}^{\pi/2}$$

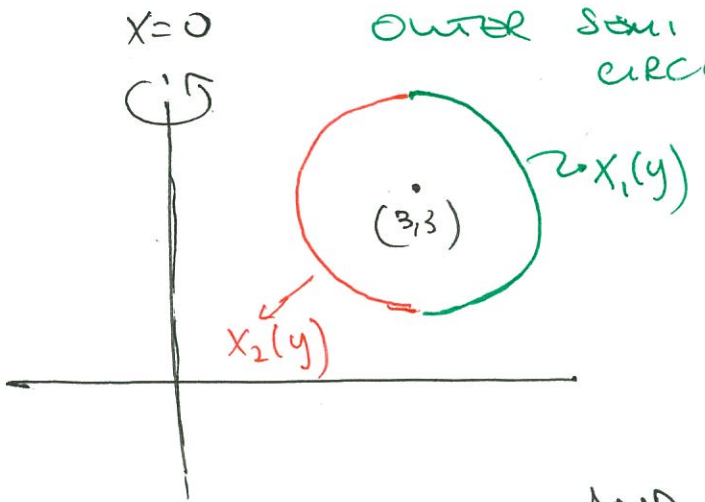
$$= 2\pi \left[\left(\frac{\pi}{2} + \frac{0}{2} \right) - \left(-\frac{\pi}{2} + \frac{0}{2} \right) \right]$$

$$\boxed{V = 2\pi^2}$$

b) FOR THE SURFACE AREA OF REVOLUTION ABOUT THE LINE $x=0$

WE GET

$$S = \underbrace{2\pi \int_2^4 x_1(y) \cdot \sqrt{1+\left(\frac{dx_1}{dy}\right)^2}\, dy}_{\text{OUTER SEMI CIRCLE}} + \underbrace{2\pi \int_2^4 x_2(y) \cdot \sqrt{1+\left(\frac{dx_2}{dy}\right)^2}\, dy}_{\text{INNER SEMI CIRCLE}}$$



WE SOLVE (1) TO OBTAIN $x_1(y)$ & $x_2(y)$

$$x_{1,2}(y) = 3 \pm \sqrt{1-(y-3)^2}$$

AND

$$\frac{dx_{1,2}}{dy} = \mp \frac{y-3}{\sqrt{1-(y-3)^2}}$$

SO THAT

$$\left(\frac{dx_{1,2}}{dy}\right)^2 = \frac{(y-3)^2}{1-(y-3)^2}$$

AND

$$1+\left(\frac{dx_{1,2}}{dy}\right)^2 = \frac{1-(y-3)^2}{1-(y-3)^2} + \frac{(y-3)^2}{1-(y-3)^2} = \frac{1}{1-(y-3)^2}$$

FINALLY

$$\sqrt{1+\left(\frac{dx_{1,2}}{dy}\right)^2} = \frac{1}{\sqrt{1-(y-3)^2}}$$

SUBSTITUTING $x_1(y)$, $x_2(y)$, $\frac{dx_1}{dy}$ AND $\frac{dx_2}{dy}$ WE GET

$$S = 2\pi\left[\int_2^4 \left(3+\sqrt{1-(y-3)^2}\right)\cdot\frac{1}{\sqrt{1-(y-3)^2}}\,dy\right.$$

$$\left. + \int_2^4 \left(3-\sqrt{1-(y-3)^2}\right)\cdot\frac{1}{\sqrt{1-(y-3)^2}}\,dy\right]$$

$$= 2\pi\left[\int_2^4 \left[\frac{3}{\sqrt{1-(y-3)^2}} + \cancel{1} + \frac{3}{\sqrt{1-(y-3)^2}} - \cancel{1}\right] dy\right]$$

$$= 12\pi\int_2^4 \frac{1}{\sqrt{1-(y-3)^2}}\,dy = 12\pi\Big[\arcsin(y-3)\Big]_2^4$$

$$= 12\pi\Big[\underbrace{\arcsin(1)}_{\pi/2} - \underbrace{\arcsin(-1)}_{(-\pi/2)}\Big]$$

$$= 12\pi\cdot[\pi] = \boxed{12\pi^2}$$

5a) $\displaystyle\int_{-3}^{2} \frac{1}{w^4}\,dw$

NOTE: INTEGRAL IS FROM $a=-3$ TO $b=2$ AND $f(w) = 1/w^4$ IS UNDEFINED @ $w=0$!!

$$= \lim_{c\to 0^-} \int_{-3}^{c} \frac{1}{w^4}\,dw + \lim_{c\to 0^+} \int_{c}^{2} \frac{1}{w^4}\,dw$$

DIVERGENT [COMPARISON WITH $\int_0^a \frac{1}{x^p}\,dx$ WITH $p \geq 1$]

b) $$\int_{-\infty}^{\infty} y e^{-y^2}\,dy = \lim_{a\to -\infty} \int_{a}^{0} y e^{-y^2}\,dy + \lim_{b\to\infty} \int_{0}^{b} y e^{-y^2}\,dy$$

USE $-y^2 = u$

$-2y\,dy = du$

$$= \lim_{a\to -\infty} \int_{-a^2}^{0} -\frac{1}{2} e^{u}\,du + \lim_{b\to\infty} \int_{0}^{-b^2} -\frac{1}{2} e^{u}\,du$$

$$= \lim_{a\to -\infty} \left[-\frac{1}{2} e^{u}\right]_{-a^2}^{0} + \lim_{b\to\infty} \left[-\frac{1}{2} e^{u}\right]_{0}^{-b^2}$$

$$= \left[-\tfrac{1}{2} + 0\right] + \left[0 - \left(-\tfrac{1}{2}\right)\right] = -\tfrac{1}{2} + \tfrac{1}{2} = 0$$

$\boxed{\text{CONVERGENT}}$

NOTE: INTEGRAND HAS ODD SYMMETRY + INTEGRATION INTERVAL IS SYMMETRIC

$\Downarrow$

$\boxed{\text{INTEGRAL} = 0}$

5 e) $$\int_{-1}^{14} \frac{1}{\sqrt[4]{z+2}}\, dz$$

NOTE: ① THE INTEGRATION INTERVAL IS FINITE $\begin{bmatrix} a=-1 \\ b=14 \end{bmatrix}$

② THE INTEGRAND, $f(z) = \frac{1}{\sqrt[4]{z+2}}$

IS FINITE FOR ALL VALUES OF z

IN THE INTEGRATION INTERVAL

$\therefore$ $\int_{-1}^{14} \frac{1}{\sqrt[4]{z+2}}\, dz$ IS CONVERGENT

$$\int_{-1}^{14} (z+2)^{-1/4}\, dz = \frac{4}{3}(z+2)^{3/4} \Big|_{-1}^{14}$$

$$= \frac{4}{3}\left[(14+2)^{3/4} - (-1+2)^{3/4}\right]$$

$$= 4/3 \left[8 - 1\right] = \frac{4}{3}(7) = \boxed{\frac{28}{3}}$$

6a)

If $\frac{dz}{du} + e^{u+z} = 0$

Then $\frac{dz}{du} = -e^{u} \cdot e^{z}$

SEPARATE VBLS

$$\frac{dz}{e^{z}} = -e^{u}\,du$$

INTEGRATE BOTH SIDES

$$\int e^{-z}\,dz = -\int e^{u}\,du$$

$$-e^{-z} = -e^{u} + C$$

$$e^{-z} = e^{u} - C$$

$$-z = \log\left[e^{u} - C\right]$$

$$\boxed{z = -\log\left[e^{u} - C\right]}$$

GENERAL SOL'N TO THE DIFFERENTIAL EQN

6b

IF $\frac{dr}{dz} = \frac{\log(z)}{rz}$ WHERE $r(1) = 2$

THEN $r\,dr = \frac{\log(z)}{z} \cdot dz$

LET $u = \log(z)$

$du = \frac{1}{z} dz$

$r\,dr = u \cdot du$

INTEGRATE BOTH SIDES

$\int r\,dr = \int u\,du$

$r^2 = u^2 + C$ BUT $u = \log(z)$

$r^2 = [\log(z)]^2 + C$

OR

$r = \pm\sqrt{[\log(z)]^2 + C}$ GENERAL SOL'N TO THE DIFFERENTIAL EQ'N

BUT WHEN $z = 1$, $r = +2$ SO

$r = +\sqrt{[\log(z)]^2 + C} \longrightarrow 2 = \sqrt{[\log(1)]^2 + C} \rightarrow C = 4$

SO $r(z) = \sqrt{[\log(z)]^2 + 4}$ PARTICULAR SOL'N TO THE DIFF EQ'N