Derivatives and limits - Revisited

We began studying limits to help us calculate the derivative of $f(x)$

A famous theorem allows us to make use of the converse

L'Hôpital's Rule

$$\text{If } \lim_{x\to a} \frac{p(x)}{q(x)} \text{ is `indeterminate' and } \lim_{x\to a} \frac{p'(x)}{q'(x)} = L, \text{ then } \lim_{x\to a} \frac{p(x)}{q(x)} = L$$

i) $\lim_{x\to a} \frac{p(x)}{q(x)}$ must be of type $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$, or other similarly expressable form

ii) $\lim_{x\to a} \frac{p'(x)}{q'(x)}$ must either exist and have a value $L < \infty$ or be of 'indeterminate' form

Note: L'Hôpital's rule can be applied repeatedly to a limit, provided that the above conditions are met.

L'Hôpital's rule - Examples

Ex: Calculate the limit $\lim_{\theta \to 0} \frac{\sin \theta}{\theta}$

Ex: Calculate the limit $\lim_{x \to \infty} \frac{3x^2 - 2x + 10}{7 - 2x^2}$

L'Hôpital's rule - Examples

Ex: Calculate the limit $\lim_{x \to 0^+} x^2 \log x$

L'Hôpital's rule - Examples

Ex: Calculate the limit $\lim_{x\to 1^+} \frac{1}{x-1} - \frac{1}{\log x}$

L'Hôpital's rule - Examples

Ex: Calculate the limit $\lim_{x \to 0^+} (e^x + x)^{1/x}$

L'Hôpital's rule - Examples

Ex: Calculate the limit $\lim_{x \to 0^+} (1 + {}^1\!/\!_x)^x$

Using Derivatives

Derivatives tell us a lot about the behaviour of functions

Appear under different guises in a variety of fields

Physics: distance $\rightarrow$ velocity $\rightarrow$ acceleration, radioactive decay
Chemistry: reaction rates, thermodynamics
Biology: population growth, predator-prey systems
Economics: profit/cost optimization

Using derivatives, we can identify key points in these functions!

Curve Sketching

Derivatives determine the 'shape' of a function

If $f'(x) > 0$, the function is increasing

If $f'(x) < 0$, the function is decreasing

If $f'(x) = 0$, the function is 'flat'

Curve Sketching - Steps

We can follow a set of steps to obtain a plot of a function $f(x)$

1. Determine the **domain** of $f(x)$
2. Determine the **intercepts** of $f(x)$ – $f(0)$ and $f(x) = 0$
3. Determine any **symmetry** and **periodicity** in $f(x)$
4. Determine any **asymptotes** and **holes** in $f(x)$
5. Determine the **critical points** of $f(x)$
6. Determine the **concavity regions** of $f(x)$
7. Sketch $f(x)$!

Asymptotoes

We find:

Vertical asymptotes in $f(x)$ at $x = a$ if $\lim_{x \to a} f(x) \to \pm\infty$

If $f(x) = \frac{p(x)}{q(x)}$, V.A. occurs at $x = a$ if $q(a) = 0$ [but $p(a) \neq 0$]

Horizontal asymtotes are given by

$$\lim_{x \to \infty} f(x) = L_1 \text{ and } \lim_{x \to -\infty} f(x) = L_2$$

Oblique asymptote occurs whenever

$$\lim_{x \to \infty} f(x) \to \pm\infty \quad \text{or} \quad \lim_{x \to -\infty} f(x) \to \pm\infty$$

We find horizontal asymtotes in $f(x) = \frac{p(x)}{q(x)}$ whenever $\deg(p) \leq \deg(q)$

We find oblique asymtotes in $f(x) = \frac{p(x)}{q(x)}$ whenever $\deg(p) > \deg(q)$

Use long division to get the equation of the oblique asymptote

Asymptotes - Examples

Ex: Find the asymptotes in $f(x) = \dfrac{2x^2}{x+1}$

Asymptotes - Examples

Ex: Find the asymptote in $f(x) = \dfrac{x^3 - x + 3}{x^2 + x - 2}$

Critical points

A function $f(x)$ has a **critical point** at $x = a$ if $a \in D_f$ and

either 1) $f'(a)$ does not exist or 2) $f'(a) = 0$

Critical points of $f(x)$ **may** determine the extreme values of $f(x)$

We classify **extreme values** of $f(x)$ as either:

1) maximum or minimum and
2) absolute/global or local

Critical points

If $f(a)$ is an extreme value, then $x = a$ **must** be a critical point

Critical points

Notice that the converse of the above statement is not always true!

The point $x = a$ may be a critical point without $f(a)$ being extreme

Stronger statment:

If $f(x)$ is continuous on a **closed interval**, it reaches its max/min!

Inflection and turning points

If $S = \{a_0, a_1, a_2, \ldots, a_n\}$ are all of the critical points of $f(x)$ with $a_i < a_{i+1}$, then $f(x)$ must have a non-zero slope for all $x \in (a_i, a_{i+1})$

Between any two critical points, $f(x)$ is either always increasing or decreasing

First Derivative Test

If $f'(a) = 0$ and $f(a)$ is an extreme value, then $x = a$ is a **turning point**

$f'(x)$ must switch signs across a turning point

If $f'(a) = 0$ but $f(a)$ is not extreme, then $x = a$ is an **inflection point**

$f'(x)$ keeps the same sign on either side of an inflection point

Concavity

Even when we know $f'(x) \neq 0$, the 'shape' of $f(x)$ is still undetermined

The second derivative determines the curvature and inflection points

If $f''(x) > 0$, then $f(x)$ is **concave up**
If $f''(x) < 0$, then $f(x)$ is **concave down**
Inflection points occur whenever the curvature changes sign

Second Derivative Test

Another way of identifying nature of extreme points

If $f'(a) = 0$ and $f''(a) < 0$, then $x = a$ is a maximum
If $f'(a) = 0$ and $f''(a) > 0$, then $x = a$ is a minimum
If $f'(a) = 0$ and $f''(a) = 0$, then the test is inconclusive

Curve sketching - Examples

Ex: Sketch the curve of $f(x) = x^3 - 4x$

Curve sketching - Examples

Ex: Sketch the curve of $f(x) = 8x^5 - 5x^4 - 20x^3$

Curve sketching - Examples

Ex: Sketch the curve of $g(w) = \dfrac{w^2}{1 - w^2}$

Curve sketching - Examples

Curve sketching - Examples

Ex: Sketch the curve of $f(x) = e^{1/x}$

Curve sketching - Examples