Sigma Notation

The Riemann sum approximation leads to expressions of long sums

To simplify the process we introduce the *sigma* (Σ) notation

Sigma Notation - Terms and Indices

Sigma notation allows for a great deal of variety and flexibility

Ex: Write down the sum of the first 10 odd integers in 3 different ways

Rules for Finite Sums

Using the usual rules of algebra, we get the following rules for sums:

1. $$\sum_{k=1}^{n} 1 = n$$

2. $$\sum_{k=1}^{n} (c \cdot a_k) = c \cdot \sum_{k=1}^{n} a_k$$

3. $$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

4. $$\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

Useful Sums

Several types of sums deserve special mention

Ex: Sum the positive integers up to a) 5 b) 7 c) 10

Can we find a general formula?

Useful Sums

Using a different approach (mathematical induction) we get similar expressions for the sum of squares and cubes

1. $$\sum_{k=1}^{n} k = 1 + 2 + 3 + 4 + 5 + \ldots + n = \frac{n(n+1)}{2}$$

2. $$\sum_{k=1}^{n} k^2 = 1 + 4 + 9 + 16 + 25 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

3. $$\sum_{k=1}^{n} k^3 = 1 + 8 + 27 + 64 + 125 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Integrals - Definite Integral

The area bounded by $f(x)$, the x-axis, and the lines $x = a$ and $x = b$

If this limit exists, we say that $f(x)$ is **integrable** on $a \leq x \leq b$

We will find that $f(x)$ is integrable on $a \leq x \leq b$ whenever:

- $f(x)$ is continuous on $a \leq x \leq b$
- $f(x)$ has a finite number of jump-discontinuities in $a \leq x \leq b$

The above gives a useful relation for breaking up integrals!

Summation Examples

Ex: If $I_1 = \int_0^4 r(q)\mathrm{d}q = 10$, $I_2 = \int_2^4 r(m)\mathrm{d}m = 1$, $I_3 = \int_4^9 r(x)\mathrm{d}x = -2$ calculate the following:

a) $\int_2^9 r(q)\mathrm{d}q$

b) $\int_0^9 r(z)\mathrm{d}z$

Integrals - Definite Integral

If $f(x) > 0$, we interpret the integral as the area under the curve

What happens if $f(x) < 0$?

What if $a > b$ in our integral?

Using the previous results we can check what happens when $b = a$

Integrals - Properties

These results allow us to write the following relations for integrals:

For any $f(x)$ and $g(x)$ that are continuous on $a \leq x \leq b$ and a constant c we have that:

1. $$\int_a^b c \, \mathrm{d}x = c \cdot (b - a)$$

2. $$\int_a^b [f(x) + g(x)] \, \mathrm{d}x = \int_a^b f(x) \, \mathrm{d}x + \int_a^b g(x) \, \mathrm{d}x$$

3. $$\int_a^b c f(x) \, \mathrm{d}x = c \cdot \int_a^b f(x) \, \mathrm{d}x$$

4. $$\int_a^b [f(x) - g(x)] \, \mathrm{d}x = \int_a^b f(x) \, \mathrm{d}x - \int_a^b g(x) \, \mathrm{d}x$$

5. If $m \leq f(x) \leq M$, then

$$m \cdot (b - a) \leq \int_a^b f(x) \, \mathrm{d}x \leq M \cdot (b - a)$$

Integral Examples

Ex: If $I_1 = \int_0^4 r(q)\mathrm{d}q = 10$, $I_2 = \int_2^4 r(m)\mathrm{d}m = 1$,
$I_3 = \int_4^9 r(x)\mathrm{d}x = -2$ and $I_4 = \int_0^9 p(y)\mathrm{d}y = 3$ calculate the following:

a) $\int_0^2 r(q)\mathrm{d}q$

b) $\int_0^9 \left[9r(z) - 4p(z)\right]\mathrm{d}z$

Average Values

The average value of an integral is defined as $av(f) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$

Interpretation: height of a rectangle with the same area as $\int_a^b f(t)\mathrm{d}t$

Ex: Calculate the average value of $\int_{-2}^{2} \sqrt{4-r^2}\mathrm{d}r$