Max/Min

Often in the sciences, we are interested in the extereme values

Key words: most, least, fastest, slowest, largest, smallest, *etc*

Focus on the critical points of $f(x)$ and extract the required data

Recall, every function that is continuous on $[a, b]$ reaches it maximum and minimum values at $x = c$ and $x = d$, respectively, with $\{c, d\} \in [a, b]$

If c is a max/min then either $f'(c) = 0$ or $f'(c)$ DNE.

Ex: Find the extreme values for $f(x) = 5 + 54x - 2x^3$ when $x \in [0, 4]$

Max/Min

Ex: Find the extreme values for $f(x) = x - \ln x$ when $x \in [\frac{1}{2}, 2]$

Max/Min

Ex: If $f(x) = \begin{cases} 4 - x^2 & \text{if } -2 \leq x < 1 \\ 2x - 1 & \text{if } \quad 1 \leq x \leq 3 \end{cases}$,

determine the absolute max and absolute min for $f(x)$

Max/Min

Ex: Find the extreme values for:

$$f(x) = x^4 + 4x^3 - 8x^2 + 20 \text{ when } x \in [0, 2]$$

Derivatives in Physics Problems

Knowing the shape of a function helps greatly in many science problems

Ex: Physics: distance $\rightarrow$ velocity $\rightarrow$ acceleration

If $s(t) = t^3 - 6t^2 + 9t$ is the position of a particle at time t, calculate:

a) The velocity at a general time t, after 2 seconds and 4 seconds

b) The times that the particle is at rest

c) The times that the particle is moving forward

d) During the first 5 seconds, when is the particle furthest from the origin

e) The accelleration at a general time t and after 2 seconds

Derivatives in Physics Problems

Derivatives in Chemistry Problems

Ex: Chemistry - Lennard-Jones (6-12) Potential

The potential energy stored in a chemical bond is modelled by:

$$u(r) = \frac{1}{r^{12}} - \frac{1}{r^6}$$

where r measures the separation between the centers of the two atoms, in Ångstroms. The equillibrium position of the bond is located at the minimum in the potential energy function.

Derivatives in Economics

Ex: Economics - Minimizing Cost, Maximizing Revenue

The cost of producing n thousand units is given by

$$C(n) = 62 - 60n + 21n^2 - 2n^3, \text{ up to 5000 units}$$

How many units should be produced to minimize the cost?

Rolle's Theorem

If $f(x)$ is a function such that

1) $f(x)$ is continuous on $x \in [a, b]$
2) $f'(x)$ exists for all $x \in (a, b)$
3) $f(a) = f(b)$

then there is a number $c \in (a, b)$ such that $f'(c) = 0$

Mean Value Theorem (MVT)

A generalization of Rolle's theorem (keep the endpoints unspecified)!

If $f(x)$ is a function such that

1) $f(x)$ is continuous on $x \in [a, b]$
2) $f'(x)$ exists for all $x \in (a, b)$

then there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

MVT

Ex: Find all the numbers c which satisfy the MVT for $f(x) = x^3 - 3x + 2$ for $a = -2$ and $b = 2$.

MVT

An implication of the MVT tells us that:

$$\text{If } f(x) \text{ is continuous on } [a,b] \text{ and } f'(x) = 0 \text{ for all } x \in [a,b],$$
$$\text{then } f(x) = K \text{ for some constant } K.$$

The only function with $f'(x) = 0$ everywhere is the constant function!

Optimization Problems

The methods of finding extreme points are very useful whenever an everyday problem can be translated into a mathematical equation.

In many real-world situations, we have to deal with constraints

Main Task: Take a word problem and relate the data given to a function!

Ex: What is the largest possible area enclosed by a rectangular pen if the total amount of fencing is 2400 m?

Optimization Problems

Optimization Problems

Ex: Repeat the previous problem, where one side of the pen is a straight river (no fencing needed).

Optimization Problems

Ex: A rectangle is inscribed in a half-circle with radius 10. Find the dimensions of the rectangle which maximize its area.

Optimization Problems

Optimization Problems

Ex: A right-circular cone is inscribed in a sphere of radius 4 m. Determine the volume of the largest such cone.

Optimization Problems

Optimization Problems

Ex: You are designing a poster that must contain 60 cm^2 of printed area. The poster is printed on paper such that there is a 4 cm border on the sides and a 3 cm border on the top and bottom. Determine the dimensions of the poster which requires the least amount of paper.

Optimization Problems

Optimization Problems

Ex: An open-top metal can is designed to contain 400mL. Find the dimensions of the can that minimize the cost of the material.

Optimization Problems

Optimization Problems

Suppose you leave point A to reach point B, 9 km away from your starting position on the opposite bank of a 3 km wide river.

You can swim at a speed of 3km/h and run at a speed of 6km/h.

Assuming the speed of the river flow is negligible, to which point should you swim, in order to minimize the travel time?
(Assume that each portion of travel occurs in a straight line)

Optimization Problems