$f(x) = a^x$ and $g(x) = \log_a x$ - Revisited

We saw from first principles that if $f(x) = a^x$

$$\begin{aligned} f'(x) &= a^x \lim_{h \to 0} \frac{a^h - 1}{h} \\ &= f(x) \cdot f'(0) \end{aligned}$$

This led to the definition of a special base, e, so that

$$f'(x) = e^x \quad \text{when} \quad f(x) = e^x$$

What about other values of a?

Need to figure out the numerical value of $f'(0)$!

Derivative of $f(x) = a^x$ - Revisited

Start with $f(x) = a^x$ use log laws:

Derivative of $g(x) = \log_a x$ - Revisited

Using implicit differentiation on $g(x) = \log_a x$ with the previous result:

Inverse trig functions

Defined in terms of inverse operations

These are functions which **undo** the original trig functions

$$\begin{aligned} \sin\theta &= a \quad \Rightarrow \quad \arcsin a = \theta \\ \cos\theta &= b \quad \Rightarrow \quad \arccos b = \theta \\ \tan\theta &= c \quad \Rightarrow \quad \arctan c = \theta \end{aligned}$$

The original functions are periodic

Must restrict the domains of the original functions

Select values of θ that result in a one-to-one function

Inverse trig functions – $f(x) = \arccos x$

Start with $g(x) = \cos x$

Look at $0 \leq x < \pi$ and note that $-1 \leq \cos x < 1$

The resulting inverse, $f(x) = \arccos(x)$ is the reflection in the line $y = x$

$D_f = \{x | x \in \mathbb{R}, -1 \leq x < 1\}$ and $R_f = \{y | y \in \mathbb{R}, 0 \leq x < \pi\}$

Inverse trig functions – $f(x) = \arcsin x$

Start with $g(x) = \sin x$

Look at $-\frac{\pi}{2} \leq x < \frac{\pi}{2}$ and note that $-1 \leq \sin x < 1$

The resulting inverse, $f(x) = \arcsin(x)$ is the reflection in the line $y = x$

$D_f = \{x | x \in \mathbb{R}, -1 \leq x < 1\}$ and $R_f = \{y | y \in \mathbb{R}, -\frac{\pi}{2} \leq x < \frac{\pi}{2}\}$

Inverse trig functions – $f(x) = \arctan x$

Start with $g(x) = \tan x$

Look at $-\frac{\pi}{2} \leq x < \frac{\pi}{2}$ and note that $\tan x$ takes on all real values

The resulting inverse, $f(x) = \arctan(x)$ is the reflection in the line $y = x$

$D_f = \{x | x \in \mathbb{R}\}$ and $R_f = \{y | y \in \mathbb{R}, -\frac{\pi}{2} \leq x < \frac{\pi}{2}\}$

Inverse trig functions - Derivatives

In each case, we obtain the derivative via implicit differentiation

Ex: Find $\frac{df}{dz}$ for $f(z) = \arcsin z$

Inverse trig functions - Derivatives

Ex: Find $\frac{df}{dz}$ for $f(z) = \arccos z$

Inverse trig functions - Derivatives

Ex: Find $\frac{df}{dz}$ for $f(z) = \arctan z$

Derivatives Overview

Derivative Rules:

$$[f(x)+g(x)]' = f'(x)+g'(x) \qquad [f(x)\cdot g(x)]' = f'(x)g(x)+f(x)g'(x)$$

$$[f(g(x))]' = f'(g(x))g'(x) \qquad \left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$$

$f(x)$	$\frac{df}{dx}$	$f(x)$	$\frac{df}{dx}$
$a_k x^k$	$k a_k x^{k-1}$	–	–
$\sin x$	$\cos x$	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos x$	$-\sin x$	$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan x$	$\frac{1}{\cos^2 x}$	$\arctan x$	$\frac{1}{1+x^2}$
a^x	$a^x \ln a$	$\log_a x$	$\frac{1}{\ln a}\frac{1}{x}$

Examples

Ex: Find the derivatives of a) $f(x) = \sin^2(x)$ and b) $g(x) = \sin(x^2)$

Examples

Ex: Find $\frac{dk}{d\theta}$ if $k(\theta) = \arctan^2(e^{3\theta})$

Examples

Ex: Find $q'(x)$ if $q(x) = x \log_4 \sin x^4$