Derivatives

Recall the graphical idea behind the derivative for non-linear functions

We obtain the derivative as the slope of the limiting sequence of secants!

Secants vs Tangents

Secants and tangents have a nice 'physical' interpretation

A **secant** is a straight line that connects two points on a curve

Represents the *average rate of change* between $x = a$ and $x = b$

A **tangent** is a secant for which the points $x = a$ and $x = b$ are *infinitesimally* close

Represents the *instantaneous rate of change* at $x = a$

Ex: Riding in a car from Toronto to Athens

Differentiability

We say that a function is *differentiable* at $x = a$ if the derivative at $x = a$ exists.

Graphically, this implies that close to $x = a$, the function can be approximated by a straight line with slope equal to $f'(a)$

NOTE: Every differential function is continuous, but not all continuous functions are differentiable!

Differentiability

Some examples of functions which are not differentiable at some point

Derivatives – First Principles

From our definition, we can calculate the derivative for some simple functions

1) Try for $f(x) = x$

Derivatives – First Principles

From our definition, we can calculate the derivative for some simple functions

1) Try for $f(x) = x^2$

Derivatives – $f'(a)$ vs $f'(x)$

We can express the 'derivative limit' in two ways:

Each form has its own utility!

Derivative notation:

Derivatives – Distributive Law

For any $f(x)$ and $g(x)$ which are differentiable at $x = a$ we have

$$(f+g)'(a) = f'(a) + g'(a)$$
$$(f-g)'(a) = f'(a) - g'(a)$$

Furthermore, for any constant c we get

$$c' = 0$$
$$(cf)'(a) = cf'(a)$$

We can combining these properties with our work from first principles, to calculate the derivative of any polynomial.

Derivatives - Exponential - $f(x) = a^x$

From first-principles we get:

Define the base of the natural logarithm, $e = 2.781828...$:

Derivatives - Trigonometric functions

We begin from first principles and use trig identities:

Derivatives - Trigonometric functions

Need the following (important) results to get the trigonometric derivatives:

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \qquad \text{and} \qquad \lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0$$

With these, we get:

$$[\sin\theta]' = \qquad \text{and} \qquad [\cos\theta]' =$$

Using these, along with the product, chain and quotient rules, we can get derivatives of all of the other trig functions

Limits and derivatives of Trig functions

Ex: $\lim_{x \to 0} \dfrac{3x}{\tan 4x}$

Derivatives - Product rule

Unlike the distributive law, $[f(x)g(x)]' \neq f'(x)g'(x)$
Check:

Instead, the *product rule* is given by

$$[f(x)g(x)]' = f(x)g'(x) + f'(x)g(x)$$

Check:

Example - Product rule

Ex: $p(x) = x^2 e^x$

Ex: $p(x) = \sqrt{x} e^x$

Example - Product rule

Ex: $[e^x \cos(x)]'$

Derivatives - Chain rule

What happens when we have to calculate the derivative of $h(x) = f(g(x))$?

For example, if $r(x) = (x^2 - 2)^3$ or $q(x) = \sqrt{e^x - 1}$

Derivatives - Chain rule

The *chain rule* is given by

$$[f(g(x))]' = f'(g(x))g'(x)$$

Check:

Example - Chain rule

Ex: $t(x) = x^2 e^{-x}$

Derivatives - Quotient rule

The last example suggests we can combine the product and chain rule for cases like $\left[\frac{f(x)}{g(x)}\right]'$

The *quotient rule* is given by

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Example - Quotient rule

Ex: $f(x) = x^2/(x-3)$

Ex: $f(x) = e^x/\sqrt{x}$

Implicit differentiation

In all of the previous examples, we have looked for $f'(x)$ with explicit expressions for $f(x)$

Ex: Find the derivative of $x^2 + y^2 = 25$ when $x = 3$.
Case I:

Implicit differentiation

Ex: Find the derivative of $x^2 + y^2 = 25$ when $x = 3$.
Case II:

Implicit differentiation

In general, we start with an implicit equaiton of the form

$$F[x, y(x)] = G[x, y(x)] \tag{1}$$

We differentiate both sides of the above equation w.r.t. x

REMEMBER: y is a funciton of x - $y(x)$

Solve the resulting equation for $y' = y'(x, y)$

Solve (1) for y to write $y' = y'(x)$, if possible

Implicit differentiation - Example

Ex: $yx^5 = 2$

Implicit differentiation - Example

Ex: $e^{\frac{x}{y}} = x - y$

Derivatives - $f(x) = \log_a(x)$

Recall $f(x) = \log_a(x)$ is the inverse of $g(x) = a^x$

Use implicit differentiation to calculate $[\log_a(x)]'$

Whenever $a = e = 2.781828...$, we get:

Derivatives - $f(x) = \log_a(x)$

Ex: Find the derivative of $f(x) = \log_{10} 3x$.

Ex: Find the derivative of $f(x) = \ln \sin^2 x$.

Higher order derivatives

Starting with $f(x)$, we can calculate the first derivative, $f'(x)$

By taking $g(x) = f'(x)$, we can also calculate $g'(x)$

The result is called the *second order derivative* of $f(x)$ – $f''(x)$

Other notation:

In this way, we can define the n-th order derivative

Higher order derivatives - Examples

Ex: If $f(x) = -x^6 + 5x^4 + x - \dfrac{2}{x}$, find $f'''(x)$:

Higher order derivatives - Examples

Ex: If $g(x) = \cos(x)$, find $g^{(33)}(x)$: