Calculus

Calculus - the study of change, as related to functions

Formally co-developed around the 1660's by Newton and Leibniz

Two main branches - differential and integral

Central role in much of modern science

Physics, especially kinematics and electrodynamics

Economics, engineering, medicine, chemistry, *etc.*

Functions - Review

If calculus is a new language, functions are its words!

Must be ‘***fluent***’ in elementary functions

1) polynomials

Functions - Review

If calculus is a new language, functions are its words!

Must be **'*fluent*'** in elementary functions

2) trigonometric

Functions - Review

If calculus is a new language, functions are its words!

Must be '***fluent***' in elementary functions

3) exponential

Functions - Review

If calculus is a new language, functions are its words!

Must be ‘***fluent***’ in elementary functions

4) logarithmic

Functions - Review

If calculus is a new language, functions are its words!

Must be ‘***fluent***’ in elementary functions

5) fractional

Differential Calculus

Differential calculus - the study of the 'shape' of curves

Qualitative descriptors are easy

Unfortunately, this is often ambiguous

Shape of a curve - Linear

Quantitative description

How **fast** is a linear function increasing/decreasing?

A:

Shape of a curve - Non-linear

Quantitative description

How **fast** is a non-linear function increasing/decreasing?

A:

Approximation improves when $x_1 = a$ and $x_2 = b$ are 'close'

Limits

We say that:

The limit of $f(x)$ at $x = a$ is equal to L
if $f(x)$ is 'close' to L whenever x is 'close' to a

(*) $\boxed{\text{Notation: } \lim_{x \to a} f(x) = L}$ (*)

1. The limit of a function *must* be finite
2. $\lim_{x \to a} f(x) = L$ tells us nothing about $f(a)$

Ex 1: $f(x) = x^2 + 4$

Limits

Ex 2: $g(x) = \frac{1}{(x+3)^2}$

Limits

Ex 3: $h(x) = \begin{cases} x^2 + 4 & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{cases}$

One-sided Limits

We say that the limit at $x = a$ **exist** and **equals** L if and only if

1. $\lim_{x \to a^-} f(x) = L$ (limit from below or left)

2. $\lim_{x \to a^+} f(x) = L$ (limit from above or right)

In that case we write

$$\circledast \quad \boxed{\lim_{x \to a} f(x) = L} \quad \circledast$$

Limit Laws

Given $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist along with a constant c, we have:

1. $\lim_{x \to a} c = c$

2. $\lim_{x \to a} x = a$

3. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

4. $\lim_{x \to a} c \cdot f(x) = c \cdot \lim_{x \to a} f(x)$

5. $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

6. $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided that $\lim_{x \to a} g(x) \neq 0$

Limits - Examples

Ex 4: $f(x) = \begin{cases} x^2 + 4 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

Limits - Examples

Ex 5: $f(x) = |x^2 - 5x + 6|$

Limits - Examples

Ex 6: $f(x) = \lfloor x \rfloor$ (we call $f(x)$ the greatest integer or floor function)

Limits - Examples

Ex 7: $f(x) = \frac{x^2-4}{x-2}$

Limits - Examples

Ex 8: $f(x) = \frac{4}{x-2}$

Limits - Examples

Ex 9: Evaluate $\lim_{x \to 1} \dfrac{x - 1}{\sqrt{x} - 1}$

Limits at $x = \pm\infty$

Instead of looking at $f(x)$ close to a point $x = a$ (with $x \neq a$), we ask at what happens when x gets very large (positive) or very small (negative).

Tells us about the behaviour of $f(x)$ on the "edges"

Limits - Examples

Ex 9: Evaluate $\displaystyle\lim_{x\to\infty} \frac{4x^2 - 6x + 7}{3x^2 - 1}$

Limits - Examples

Ex 10: Evaluate $\lim\limits_{x \to -\infty} \dfrac{2 - x}{2x^2 - 1}$

Limits - Examples

Ex 11: Evaluate $\lim_{x \to \infty} \dfrac{2x^3 + 3x - 1}{x^2 - 5x + 4}$

Limits - Examples

Ex 12: Evaluate $\lim_{x\to\infty} \sqrt{x^2 + 6x} - x$

Limits - Examples

Ex 13: Evaluate $\lim_{x \to -\infty} \sqrt{x^2 + 6x} - x$

Limits - Squeeze Theorem

Sometimes, we cannot calculate the actual limit for our function, $f(x)$

Utilize knowledge about some related functions, $g(x)$ and $h(x)$

Theorem: If $g(x) \leq f(x) \leq h(x)$ for all x 'close' to a (but $x \neq a$) and

1. $$\lim_{x \to a} g(x) = L$$

2. $$\lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} f(x) = L.$$

Limits - Examples

Ex 14: Evaluate $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x^2}\right)$

Continuity

A function $f(x)$ is continuous at $x = a$ if and only if

1. $\lim_{x \to a^-} f(x) = L$ (limit from below)

2. $\lim_{x \to a^+} f(x) = L$ (limit from above)

3. $f(a) = L$ (value at the point)

Substituting $x = a$ in a continuous function gives the limit near a

Polynomials, trigonometric, exponential and logarithmic functions are continuous at every interior point in their domain!

Easy limits whenever the function is continuous at $x = a$

One-sided Continuity

Using one-sided limits, we can define one-sided continuity as follows.
A function $f(x)$ is left-continuous at $x = a$ if and only if

1. $\lim_{x \to a^-} f(x) = L$ (limit from below)

2. $f(a) = L$ (value at the point)

Right-continous function can be defined analogously.
A function $f(x)$ is right-continuous at $x = a$ if and only if

1. $\lim_{x \to a^+} f(x) = L$ (limit from above)

2. $f(a) = L$ (value at the point)

Continuity Laws

If the functions $g(x)$ and $h(x)$ are continuous at $x = a$ and c is a constant, then the following functions are also continuous at $x = a$:

1. $g(a) + h(a)$
2. $c \cdot g(a)$
3. $g(a) \cdot h(a)$
4. $\frac{g(a)}{h(a)}$ provided $h(a) \neq 0$
5. $g(h(a))$ provided $g(x)$ is continuous at $x = h(a)$

Discontinuity

There are several ways a function could fail to be continuous

a) Jump discontinuity

Discontinuity

There are several ways a function could fail to be continuous

b) Infinite discontinuity

Discontinuity

There are several ways a function could fail to be continuous

c) Removable discontinuity

Continuity - Examples

Ex 15: Given that

$$f(x) = \begin{cases} \frac{x^3}{5} - 15 & \text{if } x \neq 5 \\ 7 & \text{if } x = 5 \end{cases}$$

for what values of x is $f(x)$ continuous.

Continuity - Examples

Ex 16: Given that

$$f(x) = \begin{cases} \frac{x-2}{x^2-4} & \text{if } x \neq 2 \\ \frac{1}{4} & \text{if } x = 2 \end{cases}$$

for what values of x is $f(x)$ continuous.

Continuity - Examples

Ex 17: Given that $f(x) = \dfrac{2-x}{x^2+2x}$ for what values of x is $f(x)$ continuous.

Continuity - Examples

Ex 18: Given that

$$f(x) = \begin{cases} 3x^2 + 2x - 3 & \text{if } x \leq 0 \\ k\cos(x) - e^{2x^2} & \text{if } x > 0 \end{cases}$$

for what values of k is $f(x)$ continuous everywhere.