Solutions to Study Questions

1.) Let X = length of a randomly selected shelf produced by the company

We are told $X \sim N(\mu_X = 170, \sigma_X^2 = (.625)^2)$

Problem: Find $P(168 \le X \le 171)$

$$P(168 \le X \le 171) = P\left(\frac{168-\mu_X}{\sigma_X} \le \underbrace{\frac{X-\mu_X}{\sigma_X}}_{\sim N(0,1)} \le \frac{171-\mu_X}{\sigma_X}\right)$$

$$= P\left(\frac{168-170}{.625} \le Z \le \frac{171-170}{.625}\right) \quad \text{where } Z \sim N(0,1)$$

$$= P(-3.2 \le Z \le 1.6)$$

$$= P(0 \le Z \le 3.2) + P(0 \le Z \le 1.6)$$

$$= .4993 + .4452 = \boxed{.9445}$$

2.) $X \sim N(\mu_X, \sigma_X^2 = 4)$

Given $.9772 = P(X \le 4)$

so $$.9772 = P(X \le 4) = P\left(\overbrace{\frac{X-\mu_X}{\sigma_X}}^{\sim N(0,1)} \le \frac{4-\mu_X}{\sigma_X}\right)$$

$$= P\left(Z \le \frac{4-\mu_X}{2}\right)$$

$$\Rightarrow \quad .4772 = P\left(0 \le Z \le \frac{4-\mu_X}{2}\right)$$

Solutions

2.(contd) Looking up .4772 in the body of the std normal table we find that .4772 corresponds to $z = 2.0$

$$\Rightarrow \frac{4-\mu_X}{2} = 2.0$$ solving for μ_X we get

$$\boxed{\mu_X = 0}$$

3) Given: $X \sim N(20, 4)$, $P(X \geq a) = .67$

$$.67 = P(X \geq a) = P\Big(\underbrace{\frac{X-\mu_X}{\sigma_X}}_{\sim N(0,1)} \geq \frac{a-\mu_X}{\sigma_X}\Big)$$

$$= P\Big(Z \geq \frac{a-\mu_X}{\sigma_X}\Big) = P\Big(Z \geq \frac{a-20}{2}\Big)$$

$$\Rightarrow 1-.67 = 1-P\Big(Z \geq \frac{a-20}{2}\Big)$$

$$\Rightarrow .33 = P\Big(Z \leq \underbrace{\frac{a-20}{2}}_{\text{must be negative}}\Big)$$

$$\Rightarrow .33 = P\Big(Z \geq -\frac{a-20}{2}\Big)$$

$$\Rightarrow .67 = P\Big(Z \leq -\frac{a-20}{2}\Big)$$

$$\Rightarrow .17 = P\Big(0 \leq Z \leq -\frac{a-20}{2}\Big)$$

Looking up .17 in body of std normal table, the value of z corresponding to .17 is .44

$$\Rightarrow -\frac{a-20}{2} = .44 \Rightarrow \boxed{a = 19.12}$$

Solutions

4.) $Y \sim Bin(n=36,\ p=.5)$ $\mu_Y = E(Y) = np = 18$

$\sigma_Y^2 = Var(Y) = np(1-p) = 9$

Use normal approximation to approximate $P(Y \geq 20)$

$$P(Y \geq 20) = P(Y \geq 19.5) = P\left(\frac{Y-\mu_Y}{\sigma_Y} \geq \frac{19.5-\mu_Y}{\sigma_Y}\right)$$

$$\doteq P\left(Z \geq \frac{19.5-18}{\sqrt{9}}\right) = P(Z \geq .5) = .5 - P(0 \leq Z \leq .5)$$

Where $Z \sim N(0,1)$

$$= .5 - .1915 = \boxed{.3085}$$

5.) Variance of the sample mean is given by

$$\sigma_{\bar{X}}^2 = \left(\frac{N-n}{N-1}\right)\frac{\sigma_X^2}{n}$$

Since we don't know σ_X^2 we have to estimate it with

$$S_X^2 = \frac{\sum x_i^2 - n\bar{x}^2}{n-1}$$

here, $\bar{X} = \frac{13+17+9+15+11}{5} = 13$

$$S_X^2 = \frac{13^2+17^2+9^2+15^2+11^2 - 5(13)^2}{5-1} = 10$$

So our estimate of $\sigma_{\bar{X}}^2$ is

$$\frac{N-n}{N-1}\,\frac{S_X^2}{n} = \frac{45-5}{45-1}\,\frac{10}{5} = \boxed{1.8181}$$

Solutions

6.) Given: $X_1, X_2, \ldots, X_{16} \overset{iid}{\sim} N(\mu_X = 10, \sigma_X^2 = 100)$

Find $P(\bar{X} > 4)$

$$\bar{X} \sim N\left(\mu_X = 10, \frac{\sigma_X^2}{n} = \frac{100}{16} = 6.25\right)$$

$$\Rightarrow P(\bar{X} > 4) = P\left(\frac{\bar{X} - \mu_X}{\sigma_X/\sqrt{n}} > \frac{4 - \mu_X}{\sigma_X/\sqrt{n}}\right)$$

$$= P\Big(\underbrace{Z}_{\sim N(0,1)} > \frac{4-10}{\sqrt{6.25}}\Big) = P(Z > -2.4)$$

$$= P(Z \le 2.4) = .5 + P(0 \le Z \le 2.4)$$

$$= .5 + .4918 = \boxed{.9918}$$

7.) Sample of jewels: $\cancel{X_1, X_2, \ldots, X_{36} \overset{iid}{\sim} N(\mu_X = 1, \sigma_X^2 = (.3)^2)}$

$X_1, X_2, \ldots, X_{36}$ an iid sample with common mean $\mu_X = 1$ and common variance $\sigma_X^2 = (.3)^2$
(Not told that X's are normal)

Want to find $P(\bar{X} < 1.11)$

Since sample size $n \ge 30$ ($n = 36$ here), the CLT says

$$\bar{X} \mathrel{\dot\sim} N\left(\mu_X, \underbrace{\frac{N-n}{N-1}} \frac{\sigma_X^2}{n}\right)$$

w/out replacement

Solutions

so $\bar{X} \overset{\cdot}{\sim} N\left(1, \frac{100-36}{99}\frac{(.3)^2}{36} = .001616\right)$

$$P(\bar{X} < 1.11) = P\Big(\underbrace{\frac{\bar{X}-\mu_{\bar{X}}}{\sigma_{\bar{X}}}}_{\overset{\cdot}{\sim} N(0,1)} < \frac{1.11-\mu_{\bar{X}}}{\sigma_{\bar{X}}}\Big)$$

$$\doteq P\left(Z < \frac{1.11-1.0}{\sqrt{.001616}}\right)$$

$$= P(Z < 2.74) = .5 + P(0 \le Z \le 2.74)$$

$$= .5 + .4969 = .9969$$

8) (c) is true

9) Given: $X_1, X_2, \ldots, X_{10} \overset{iid}{\sim} N(\mu_X, \sigma_X^2)$ both unknown

$\bar{X} = 83 \quad S_X^2 = 15$

Exact 90% CI given by

$$\bar{X} \pm t_{\alpha/2}(n-1)\frac{S_X}{\sqrt{n}}$$

$\alpha = 1 - .9 = .1$
$\Rightarrow \alpha/2 = .05$

or

$$83 \pm \underbrace{t_{.05}(9)}_{=1.833}\sqrt{\frac{15}{10}} = \boxed{(80.76, 85.24)}$$

Solutions

10.) a) $\mu_{\bar{X}} = E(\bar{X}) = \mu_X = 5$

$$\sigma^2_{\bar{X}} = Var(\bar{X}) = \frac{\sigma^2_X}{n} = \frac{16}{100} = .16$$

b.) $\bar{X} \sim N(5, .16)$

c) $$P(\bar{X} > 6) = P\Big(\underbrace{\frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}}}_{\sim N(0,1)} > \frac{6 - \mu_{\bar{X}}}{\sigma_{\bar{X}}}\Big) = P\Big(\overset{\sim N(0,1)}{Z} > \frac{6-5}{\sqrt{.16}}\Big)$$

$$= P(Z > 2.5) = .5 - P(0 \le Z \le 2.5)$$
$$= .5 - .4938 = .0062$$

d.) $Y = X_1 + X_2 - 2X_3$ is a linear combination of independent normals each with mean 5 and Variance 16

so $$E(Y) = E(X_1 + X_2 - 2X_3) = E(X_1) + E(X_2) + E(-2X_3)$$
$$= 5 + 5 + -2E(X_3)$$
$$= 5 + 5 - 10 = 0$$

$$Var(Y) = Var(X_1 + X_2 - 2X_3) = Var(X_1) + Var(X_2) + Var(-2X_3)$$

$$= 16 + 16 + (-2)^2 Var(X_3) = 32 + 4(16) = 96$$

e) $Y \sim N(0, 96)$

Solutions

#10 part f) Find $P(-1 < Y < 1)$

$$P(-1 < Y < 1) = P\left(\frac{-1-\mu_Y}{\sigma_Y} < \underbrace{\frac{Y-\mu_Y}{\sigma_Y}}_{\sim N(0,1)} < \frac{1-\mu_Y}{\sigma_Y}\right)$$

$$= P\left(\frac{-1-0}{\sqrt{96}} < Z < \frac{1-0}{\sqrt{96}}\right) = P(-.10 < Z < .10)$$

$$= 2P(0 \le Z \le .10) = 2(.0398)$$

$$= .0796$$

#11. $X_1, X_2, \ldots, X_{16} \overset{iid}{\sim} N(\mu_X, \sigma_X^2)$

$\bar{X} = 22$, $S_X^2 = 9$

a) point estimate of μ_X is 22

b) 95% CI for μ_X assuming $\sigma_X^2 = 10$ known

95% CI given by

$$\bar{X} \pm Z_{\alpha/2} \frac{\sigma_X}{\sqrt{n}}$$

here $\alpha = .05 \Rightarrow \frac{\alpha}{2} = .025$
$\Rightarrow Z_{\alpha/2} = 1.96$

So our CI is

$$22 \pm 1.96 \frac{\sqrt{10}}{\sqrt{16}} = 22 \pm 1.55$$

$$= (20.45, 23.55)$$

Solutions

~~Pt.~~ #11 (Continued)

part (c) Now form 95% CI when σ_X^2 unknown:

CI given by $\bar{X} \pm t_{\alpha/2}(n-1)\frac{S_X}{\sqrt{n}}$

or $22 \pm \underbrace{t_{.025}(15)}_{=2.131}\frac{3}{\sqrt{16}} = 22 \pm 1.60$

$= (20.40, 23.60)$

d.) Now form 90% CI when σ_X^2 unknown

$\bar{X} \pm t_{\alpha/2}(n-1)\frac{S_X}{\sqrt{n}}$ where now $\alpha = .1$
$\Rightarrow \frac{\alpha}{2} = .05$

$22 \pm \underbrace{t_{.05}(15)}_{=1.753}\frac{3}{\sqrt{16}} = 22 \pm 1.31 = (20.68, 23.31)$

#12. $X_1, X_2, \ldots, X_{100} \overset{iid}{\sim} N(0, 100)$

a) $P(X_1 + X_2 + \cdots + X_{100} > 100) = P\left(\frac{1}{100}(X_1 + \cdots + X_{100}) > \frac{1}{100}100\right)$

$= P(\bar{X} > 1)$ where $\bar{X} \sim N\left(0, \frac{100}{100} = 1\right)$

$= P(Z > 1) = .5 - P(0 \le Z \le 1) = .5 - .3413$
$= .1587$

Solutions

\# 12 b.) Find prob that more than 30 of the X_i's exceed 10

This is a binomial probability

Each of the 100 sample values is an independent identical trial for which there is a success $(X_i > 10)$ and a failure $(X_i \le 10)$ with probability of success $= p$ that is the same on each trial

$$p = P(\underbrace{X_i}_{\text{any of the } X_i\text{'s}} > 10) = P\Big(\underbrace{\frac{X_i - \mu_X}{\sigma_X}}_{\sim N(0,1)} > \frac{10 - \mu_X}{\sigma_X}\Big)$$

$$= P\Big(Z > \frac{10 - 0}{\sqrt{100}}\Big) = P(Z > 1) = .1587 \text{ from part (a)}$$

Let $Y = \#$ of X_i's that are > 10

$$Y \sim Bin(n = 100, p = .1587) \leftarrow \begin{array}{l} E(Y) = np = 15.87 \\ Var(Y) = np(1-p) \\ \quad = 13.35 \end{array}$$

We want $P(Y > 30) = P(Y \ge 31)$ ← hard to compute exactly so use normal approx.

$$= P(Y \ge 30.5) = P\Big(\underbrace{\frac{Y - \mu_Y}{\sigma_Y}}_{\dot{\sim} N(0,1)} \ge \frac{30.5 - \mu_Y}{\sigma_Y}\Big)$$

Solutions

#12b) so $P(Y>30) \doteq P\left(Z \geq \frac{30.5-15.87}{\sqrt{13.35}}\right) = P(Z \geq 4.00)$

$= \boxed{3.17 \times 10^{-5}}$

(essentially 0)

#13a) ~~[illegible]~~

For testing $H_0: p = p_0$ ($p_0 = .5$ in this example)

we have a choice. We can form our test based on the sample proportion $\bar{p} = \frac{X}{n}$, or based on X = # of successes where

$$X \sim Bin(n, p)$$

The 2 approaches are equivalent, so let's go with the one we're familiar with:

Under H_0, $X \sim Bin(265, .5)$

and

$$\mu_X = E(X) = 265(.5) = 132.5$$
$$\sigma_X^2 = Var(X) = 265(.5)(1-.5) = 66.25$$

Our p-value is $= 2P(X \geq 144)$ ← computed under $H_0: p = .5$

$= 2P(X \geq 143.5)$ adjusting for discreteness of X

$= 2P\left(\underbrace{\frac{X-\mu_X}{\sigma_X}}_{\dot\sim N(0,1)} \geq \frac{143.5-\mu_X}{\sigma_X}\right) \doteq 2P\left(Z \geq \frac{143.5-132.5}{}\right)$

Solutions

13 a) $\Rightarrow$ p-value $\doteq 2P(Z \geq 1.35)$

$$= 2[.5 - P(0 \leq Z \leq 1.35)]$$

$$= 2[.5 - .4115] = .1770$$

so since $p \nless \alpha = .05$ we don't reject $H_0: p = .5$

The equivalent approach based on $\bar{p}$ would be to notice

$$\bar{p} = \frac{X}{n} = \frac{144}{265} = .543$$

and since n is large $\bar{p} \,\dot{\sim}\, N\left(p, \frac{p(1-p)}{n}\right)$

Under H_0, $p = .5$ and $\frac{p(1-p)}{n} = \frac{.5(.5)}{265} = (.03071)^2$

so $\bar{p} \,\dot{\sim}\, N(.5, (.03071)^2)$

p-value $= 2P(\bar{p} \geq .543)$ But since $\bar{p}$ is discrete (Notice $\bar{p}$ can only take values $\frac{0}{265}, \frac{1}{265}, \ldots, \frac{265}{265}$) we need a continuity (or discreteness) correction. Its more accurate to compute

$$\text{p-value} = 2P\left(\bar{p} \geq .543 - \frac{1}{2n}\right) = 2P\left(\bar{p} \geq .543 - \frac{1}{2(265)}\right)$$

this would be a + if we had a $\leq$ in place of the $\geq$ here

Solutions

#13a) $\Rightarrow$ $p\text{-value} = 2P\left(\bar{p} \geq .543 - \frac{1}{2(265)}\right)$

$$= 2P\left(\frac{\bar{p} - \mu_{\bar{p}}}{\sigma_{\bar{p}}} \geq \frac{.543 - \frac{1}{2(265)} - \mu_{\bar{p}}}{\sigma_{\bar{p}}} \right)$$

when computed exactly

$$\doteq 2P\left(Z \geq \frac{.543 - \frac{1}{530} - .5}{.03071} \right) = 2P(Z \geq 1.35)$$

$$= \boxed{.1770} \text{ as before.}$$

#13b.) An approx $(1-\alpha) \times 100\%$ CI for p is given by

$$\bar{p} \pm z_{\alpha/2} \sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

For $\alpha = .1$, $z_{\alpha/2} = z_{.05} = 1.645$

So we have

$$.543 \pm 1.645 \sqrt{\frac{.543(1-.543)}{265}}$$

$$= .543 \pm .050 = (.493, .593)$$

A slightly better interval can be obtained by using a continuity correction:

$$\bar{p} \pm \left\{ z_{\alpha/2} \sqrt{\frac{\bar{p}(1-\bar{p})}{n}} + \frac{1}{2n} \right\}$$

$$\Rightarrow \quad .543 \pm \left\{ 1.645 \sqrt{\frac{.543(1-.543)}{265}} + \frac{1}{2(265)} \right\}$$

$= .543 \pm .052 = (.491, .595)$ only slightly diff't.