STAT 8260 Final Exam – SAMPLE EXAM*
SHOW ALL WORK

Name: Answer Key

1. Suppose that data for a one-way classification of three observations in each of two classes (treatment groups, for example) are as follows.

Response (y_{ij})		Covariate (x_{ij})	
Class I	Class II	Class I	Class II
34	32	2	11
36	40	8	3
47	51	5	7

Here, x_{ij} is a covariate that is measured at the same time as the response for each experimental unit ($j = 1, 2, 3$) in each class ($i = 1, 2$).

For these data, write down, in matrix and vector form, the model equation for

a. **(5 pts.)** the analysis of covariance model $y_{ij} = \mu + \alpha_i + \beta x_{ij} + \varepsilon_{ij}$, $i = 1, 2$, $j = 1, 2, 3$;

$$\begin{pmatrix} 34 \\ 36 \\ 47 \\ 32 \\ 40 \\ 51 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 8 \\ 1 & 1 & 0 & 5 \\ 1 & 0 & 1 & 11 \\ 1 & 0 & 1 & 3 \\ 1 & 0 & 1 & 7 \end{pmatrix} \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta \end{pmatrix} + \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{21} \\ \varepsilon_{22} \\ \varepsilon_{23} \end{pmatrix}$$

$$\underset{\sim}{y} = \underset{\sim}{X} \quad \underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$$

* This is really not a sample exam, but rather a collection of problems representative of what I might ask on your final. There are more questions here than I would ask you. That is, this is much longer than your exam will be. These questions should give you good practice for the actual final exam.

b. **(5 pts.)** the intra-class regression model $y_{ij} = \mu + \alpha_i + \beta_i x_{ij} + \varepsilon_{ij}$, $i = 1, 2$, $j = 1, 2, 3$.

$$\begin{pmatrix} 34 \\ 36 \\ 47 \\ 32 \\ 40 \\ 51 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 8 & 0 \\ 1 & 1 & 0 & 5 & 0 \\ 1 & 0 & 1 & 0 & 11 \\ 1 & 0 & 1 & 0 & 3 \\ 1 & 0 & 1 & 0 & 7 \end{pmatrix} \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{21} \\ \varepsilon_{22} \\ \varepsilon_{23} \end{pmatrix}$$

$$\underset{\sim}{y} = \underset{\sim}{X} \quad \underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$$

2. Let

$$\begin{aligned} y_1 &= \alpha_1 + \alpha_2 + \varepsilon_1, \\ y_2 &= \alpha_1 - \alpha_2 - \alpha_3 + \varepsilon_2, \qquad (*) \\ y_3 &= \alpha_1 + \alpha_2 + \varepsilon_3, \end{aligned}$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)^T \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_3)$. Consider the restriction $\alpha_3 = \alpha_2 - \alpha_1$.

a. **(4 pts.)** Is model (*) an overparameterized model? Why or why not?

$\underset{\sim}{X} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix} = (\underset{\sim}{x}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3)$. Notice $\underset{\sim}{x}_3 = \dfrac{\underset{\sim}{x}_1 - \underset{\sim}{x}_2}{-2}$

$\Rightarrow \underset{\sim}{x}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3$ are linearly dependent

$\Rightarrow$ Model is overparameterized and $\underset{\sim}{X}$ is not of full rank

b. **(4 pts.)** Does the restriction $\alpha_3 = \alpha_2 - \alpha_1$ correspond to a real restriction on the model space, or is this restriction a side condition? Justify your answer.

Since $\underset{\sim}{X}$ has 3 columns, but rank of only 2, any single linear restriction $\underset{\sim}{T}\underset{\sim}{\beta} = \underset{\sim}{0}$ for $\underset{\sim}{T}$ a 1×3 matrix (row vector), in this case $\underset{\sim}{T} = (-1, 1, -1)$, will identify the model as long as $\underset{\sim}{T}\underset{\sim}{\beta}$ is non-estimable. Here, $\underset{\sim}{T}\underset{\sim}{\beta} = \alpha_1 - \alpha_2 + \alpha_3$ is non-estimable (an estimable function here $\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 \alpha_3$ must satisfy $\lambda_2 - \lambda_1 = 2\lambda_3$) The reduced model here has model matrix $\tilde{\underset{\sim}{X}} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \\ 1 & 1 \end{pmatrix}$ and $C(\underset{\sim}{X}) = C(\tilde{\underset{\sim}{X}})$ so the restriction is a side condition.

c. **(6 pts.)** Obtain the constrained least-squares estimator of $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)^T$ based on model (*) subject to the restriction $\alpha_3 = \alpha_2 - \alpha_1$. Show your work.

Substituting the restriction into the model equations, we get an unrestricted, reduced model as follows

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix}, \quad \underset{\sim}{\varepsilon} \sim N(0, \sigma^2 I_3)$$

or $\quad \underset{\sim}{y} = \underset{\sim}{\tilde{X}} \underset{\sim}{\gamma} + \varepsilon$

$$\hat{\underset{\sim}{\gamma}} = (\underset{\sim}{\tilde{X}}^T \underset{\sim}{\tilde{X}})^{-1} \underset{\sim}{\tilde{X}}^T \underset{\sim}{y} \quad \text{where } \underset{\sim}{\tilde{X}}^T \underset{\sim}{\tilde{X}} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ -2 & 6 \end{pmatrix}$$

$$\Rightarrow (\tilde{X}^T \tilde{X})^{-1} = \frac{1}{36-4} \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix} = \frac{1}{32} \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}, \quad \underset{\sim}{\tilde{X}}^T \underset{\sim}{y} = \begin{pmatrix} y_1 + 2y_2 + y_3 \\ y_1 - 2y_2 + y_3 \end{pmatrix}$$

$$\Rightarrow \hat{\underset{\sim}{\gamma}} = \frac{1}{32} \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} y_1 + 2y_2 + y_3 \\ y_1 - 2y_2 + y_3 \end{pmatrix} = \frac{1}{32} \begin{pmatrix} 8y_1 + 8y_2 + 8y_3 \\ 8y_1 - 8y_2 + 8y_3 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} y_1 + y_2 + y_3 \\ y_1 - y_2 + y_3 \end{pmatrix}$$

$$\hat{\alpha}_1 = \hat{\gamma}_1 = \frac{1}{4}(y_1 + y_2 + y_3), \quad \hat{\alpha}_2 = \hat{\gamma}_2 = \frac{1}{4}(y_1 - y_2 + y_3)$$

$$\hat{\alpha}_3 = \hat{\alpha}_2 - \hat{\alpha}_1 = -\frac{1}{2} y_2$$

I used the notation T for this matrix this semester

d. **(4 pts.)** Repeat part (c) using a different method to obtain your estimator. Show your work.

$$\hat{\alpha} = (X^T X + A^T A)^{-1} X^T y$$

$$X^T X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$(X^T X + A^T A)^{-1} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$$

$$X^T y = \begin{pmatrix} y_1 + y_2 + y_3 \\ y_1 - y_2 + y_3 \\ -y_2 \end{pmatrix} \Rightarrow \hat{\alpha} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} y_1 + y_2 + y_3 \\ y_1 - y_2 + y_3 \\ -y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} y_1 + \frac{1}{4} y_2 + \frac{1}{4} y_3 \\ \frac{1}{4} y_1 - \frac{1}{4} y_2 - \frac{1}{4} y_3 \\ -\frac{1}{2} y_2 \end{pmatrix}$$

3. Consider the linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ where $\boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_n)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)^T$ and $\mathbf{X} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{pmatrix}$. For each of the following three quantities, determine whether or not it is estimable? Justify your answers.

$X = (X_1, X_2, X_3)$, where $X_1 = (1,1,1,1)^T$, $X_2 = (1,1,2,0)^T$, $X_3 = (0,0,-1,1)^T$

a. **(5 pts.)** $\eta_1 = \beta_2$.

b. **(5 pts.)** $\eta_2 = \beta_1 - \beta_2$.

c. **(5 pts.)** $\eta_3 = \beta_1 + 2\beta_2 - \beta_3$.

A contrast $\lambda^T \beta$ is estimable iff there exists an a so that $X^T a = \lambda = (\lambda_1, \lambda_2, \lambda_3)^T$, or

$$\left.\begin{aligned} X_1^T a &= \lambda_1 \\ X_2^T a &= \lambda_2 \\ X_3^T a &= \lambda_3 \end{aligned}\right\} (*)$$

But notice here that $X_1 = X_2 + X_3$. Therefore $(*)$ imply $\lambda_1 = X_2^T a + X_3^T a = \lambda_2 + \lambda_3$

In part (a) $\lambda = (0,1,0)^T$

(b) $\lambda = (1,-1,0)^T$

(c) $\lambda = (1,2,-1)^T$ ← only in (c) does $\lambda_1 = \lambda_2 + \lambda_3$ $(2-1=1)$

so η_3 is estimable, ~~η_2~~ η_1 and η_2 are not estimable.

4. Consider the two-way layout model with interaction and unbalanced replication in each treatment:

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}, \quad \begin{array}{l} i = 1,2,3 \\ j = 1,2 \\ k = 1,\ldots,n_{ij} \end{array}.$$

Let $\bar{\mu}_{i\cdot} = \mu + \alpha_i + \bar{\beta}_{\cdot} + \bar{\gamma}_{i\cdot}$ denote the mean at the i^{th} level of factor A (averaged over levels of factor B).

a. **(10 pts.)** Construct a matrix $\mathbf{A}$ and a vector $\mathbf{d}$ so that the hypothesis $H_0 : \bar{\mu}_{1\cdot} = \bar{\mu}_{2\cdot} = \bar{\mu}_{3\cdot}$ can be equivalently expressed as $H_0 : \mathbf{A}\boldsymbol{\beta} = \mathbf{d}$, for $\boldsymbol{\beta} = (\mu, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}, \gamma_{31}, \gamma_{32})^T$.

An equivalent expression for H_0 is $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \bar{\mu}_{1\cdot} \\ \bar{\mu}_{2\cdot} \\ \bar{\mu}_{3\cdot} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

or $$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \underset{\sim}{\beta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or $$\underbrace{\begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}}_{= \underset{\sim}{A}} \underset{\sim}{\beta} = \underset{\sim}{0} = \underset{\sim}{d}$$

5. **(10 pts.)** Consider the linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ where $\boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma^2\mathbf{I}_n)$ and $\mathbf{X}$ is $n \times k$ with $\text{rank}(\mathbf{X}) = r < k$. Prove the following statement: if $\mathbf{c}^T\boldsymbol{\beta}$ is estimable and $\hat{\boldsymbol{\beta}}_1$ and $\hat{\boldsymbol{\beta}}_2$ are any two least-squares estimators (i.e., two solutions to the normal equations), then $\mathbf{c}^T\hat{\boldsymbol{\beta}}_1 = \mathbf{c}^T\hat{\boldsymbol{\beta}}_2$.

$\underline{c}^T\beta$ estimable implies $\underline{c} = \underline{X}^T\underline{a}$ for some $\underline{a} \in \mathbb{R}^n$

If $\hat{\beta}_1, \hat{\beta}_2$ are two least-squares estimators, then

$\hat{\beta}_1 = \underline{G}_1 \underline{X}^T \underline{y}$, $\hat{\beta}_2 = \underline{G}_2 \underline{X}^T \underline{y}$

for $\underline{G}_1, \underline{G}_2$ two not necessarily equal generalized inverses of $\underline{X}^T\underline{X}$.

$$\Rightarrow \underline{c}^T\hat{\beta}_1 = \underline{a}^T \underline{X}\, \underline{G}_1 \underline{X}^T \underline{y} = \underline{a}^T P_{C(\underline{X})} \underline{y}$$
$$= \underline{a}^T \underline{X}\, \underline{G}_2 \underline{X}^T \underline{y} = \underline{a}^T \underline{X} \hat{\beta}_2 = \underline{c}^T \hat{\beta}_2$$

i.e., $\underline{X}\,\underline{G}_1 \underline{X}^T = \underline{X}\,\underline{G}_2 \underline{X}^T = P_{C(\underline{X})}$ is unique for any and all choices of generalized inverses of $\underline{X}^T\underline{X}$

~~2~~ 6. Consider the linear model $y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}$, $i = 1, 2$, $j = 1, 2$, or

$$\begin{pmatrix} y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{22} \end{pmatrix}.$$

a **(4 pts.)** Let $\bar{\mu}_{i\cdot}$, $i = 1, 2$, represent the mean response in treatment i averaged over j (averaged over the levels of the factor indexed by j). Write down an estimable function of the model parameters equal to the comparison $\bar{\mu}_{1\cdot} - \bar{\mu}_{2\cdot}$ of the treatment means.

$$\bar{\mu}_{i\cdot} = \frac{1}{2}\sum_{j=1}^{2} \mu_{ij} = \frac{1}{2}\sum_{j=1}^{2}(\mu + \alpha_i + \beta_j)$$

$$= \mu + \alpha_i + \bar{\beta}_\cdot, \text{ where } \bar{\beta}_\cdot = \frac{1}{2}\sum_{j=1}^{2}\beta_j$$

$$\Rightarrow \bar{\mu}_{1\cdot} - \bar{\mu}_{2\cdot} = (\mu + \alpha_1 + \bar{\beta}_\cdot) - (\mu + \alpha_2 + \bar{\beta}_\cdot)$$

$$= \boxed{\alpha_1 - \alpha_2}$$

which is estimable in this model.

For a function $\underline{c}^T\beta$ to be estimable, we must have $\underline{c} = \underline{X}^T\underline{a}$ for some $\underline{a}$. I.e., if $\underline{x}_i$ is the i^{th} col. of $\underline{X}$, then must have $c_1 = \underline{x}_1^T\underline{a}$, $c_2 = \underline{x}_2^T\underline{a}$, $c_3 = \underline{x}_3^T\underline{a}$, $c_4 = \underline{x}_4^T\underline{a}$, $c_5 = \underline{x}_5^T\underline{a}$

but notice that $\underline{x}_3 = \underline{x}_1 - \underline{x}_2$ and $\underline{x}_5 = \underline{x}_1 - \underline{x}_4$

so must have
$$c_3 = \underline{x}_1^T\underline{a} - \underline{x}_2^T\underline{a} = c_1 - c_2$$
$$c_5 = \underline{x}_1^T\underline{a} - \underline{x}_4^T\underline{a} = c_1 - c_4$$

$\alpha_1 - \alpha_2$ corresponds to $\underline{c} = (0, 1, -1, 0, 0)^T$
which satisfies $c_3 = c_1 - c_2$ $(-1 = 0 - 1)$
and $c_5 = c_1 - c_4$ $(0 = 0 - 0)$

$\Rightarrow \alpha_1 - \alpha_2$ estimable.

7.

For each of the following hypotheses, determine whether or not it is testable? Justify your answers.

b. **(4 pts.)** $H_1 : \alpha_1 - \beta_1 = 0.$ $\underset{\sim}{A}_1 = (0, 1, 0, -1, 0)$

c. **(4 pts.)** $H_2 : \alpha_1 - \alpha_2 + \beta_1 - \beta_2 = 0.$ $\underset{\sim}{A}_2 = (0, 1, -1, 1, -1)$

d. **(4 pts.)** $H_3 : \begin{pmatrix} \alpha_1 - \alpha_2 \\ \beta_1 - \beta_2 \\ \alpha_1 - \alpha_2 + \beta_1 - \beta_2 \end{pmatrix} = 0.$ $\underset{\sim}{A}_3 = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 & -1 \end{pmatrix}$

~~Reason~~ Must have 1.) $\underset{\sim}{A}\beta$ estimable
2.) $\underset{\sim}{A}$ have full row rank
3.) $\underset{\sim}{A}\beta$ have no more than rank$(\underset{\sim}{X})$ elements

rank$(\cancel{A}\underset{\sim}{X}) = 3$ so all three $\underset{\sim}{A}_i$'s satisfy (3).

rank$(A_1) = 1$, rank$(A_2) = 1$, rank$(A_3) = 2 < \#$ rows

$\Rightarrow$ by criterion (2) H_3 is not testable

for estimability each row of $\underset{\sim}{A}_i$ matrices must satisfy $c_3 = c_1 - c_2$, $c_5 = c_1 - c_4$ (see part a)

this is true for $\underset{\sim}{A}_2$, but not for $\underset{\sim}{A}_1$

$\Rightarrow$ ~~A~~ H_2 is only testable hypothesis

(H_1 involves $A\beta$ ~~matrix~~ w/ nonestimable elements
H_3 involves $\underset{\sim}{A}$ matrix w/ less than full rank)

7. An experiment was conducted in which 12 experimental units were randomly divided into 3 treatment groups of 4 units each. The three treatments were then applied and measurements of the response made. The one-way ANOVA model

$$y_{ij} = \mu + \alpha_i + \varepsilon_{ij}, \quad i = 1,2,3, \quad j = 1,\ldots,4, \quad \{\varepsilon_{ij}\} \overset{iid}{\sim} N(0,\sigma^2)$$

was used to analyze the data and the following results were obtained:

$$(\mathbf{X}^T\mathbf{X})^- = \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & 2 & 1 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \hat{\beta} = \begin{pmatrix} \hat{\mu} \\ \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\alpha}_3 \end{pmatrix} = \begin{pmatrix} 72.50 \\ 7.50 \\ 22.25 \\ 0.00 \end{pmatrix} \quad SS_E = 1669.75$$

b. **(8 points)** The sample variance of the data is $\frac{1}{11}\sum_{i=1}^{3}\sum_{j=1}^{4}(y_{ij} - \bar{y}_{..})^2 = 244.99$. Test the hypothesis of equal treatment means $H_0 : \mu_1 = \mu_2 = \mu_3$ using a significance level of 0.05.

$$F = \frac{(SSE_0 - SSE)/(dfE_0 - dfE)}{SSE/dfE}$$

The null model here is $y_{ij} = \mu + \varepsilon_{ij}$.
Which as $MSE_0 \doteq$ sample variance $= 244.99$

$\Rightarrow SSE_0 = 11(244.99) = 2694.89$

$df_{E_0} = 11, \quad dfE = n - \text{rank}(X) = 12 - 3 = 9$

$$\Rightarrow F = \frac{(2694.89 - 1669.75)/(11-9)}{1669.75/9}$$

$= 2.763$

$F_{2,9,.95} = 4.26 \quad \Rightarrow$ fail to reject H_0

8. A Latin square design is an experimental design in which experimental units are cross-classified by two blocking factors, and then one treatment is observed in each combination of the blocking factors. For example, the following two tables display the design (first square) and data (second square), from a Latin square design with two levels of each blocking factor (rows and columns) and two levels of the treatment factor (the letters).

A	B
B	A

,

y_{111}	y_{122}
y_{212}	y_{221}

All combinations of the levels of the blocking factors (represented by the rows and columns) are observed, but only one of the two treatments is observed in each row/column combination. Here, y_{ijk} represents the observation of treatment k (if there is one) in row i and column j. Note that the data consist of just four observations $\mathbf{y} = (y_{111}, y_{122}, y_{212}, y_{221})^T$.

a. **(8 points)** Suppose that we consider all block and treatment effects to be fixed and use the linear model

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + \varepsilon_{ijk},$$

for all i, j, k combinations observed in ther data. We make the usual spherical variance-covariance assumption on the errors. Write down the model matrix $\mathbf{X}$ and parameter vector $\boldsymbol{\beta}$ for this situation. What is rank($\mathbf{X}$) here?

$$\underline{\beta} = (\mu, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2)^T$$

$$\underline{X} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$\underline{j}_4$ sum to $\underline{j}_4$

$\text{rank}(\underline{X}) = 4 = \text{nrows}(\underline{X})$

b. **(5 points)** Write down an appropriate set of side conditions for reparameterizing this model as a full rank model.

$$\alpha_2 = 0, \ \beta_2 = 0, \ \gamma_2 = 0$$

c. **(5 points)** Write down the reparameterized model matrix and parameter vector based on your side conditions from part (b).

$$\tilde{\beta} = (\mu, \alpha_1, \beta_1, \gamma_1)^T$$

$$\tilde{X} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

d. **(6 points)** Compute SSE for data from this design analyzed using the model given above.

Since the full rank parameterization of this model has 4 parameters and 4 observations, the model should fit perfectly. I.e., $\hat{y} = P(y \mid C(\underline{X})) = \underline{y}$ here, so $SSE = (y-\hat{y})^T(y-\hat{y}) = \underline{0}^T\underline{0} = 0$.

9. Consider the effects version of the one-way layout model

$$y_{ij} = \mu + \alpha_i + \varepsilon_{ij}, \quad \{\varepsilon_{ij}\} \overset{iid}{\sim} N(0, \sigma^2)$$

for $i = 1, 2, 3$ groups, and $j = 1, 2$ observations in each group.

a. **(7 points) Consider the hypothesis $H_0 : \alpha_1 + \alpha_3 = 2\alpha_2$. Is this hypothesis testable? Justify your answer.**

$\underline{X} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

H_0: $\underline{c}^T\underline{\beta} = 0$ for $\underline{c} = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}$, $\underline{\beta} = \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$

$c^T\beta$ is estimable here. Why?: For $\underline{c}^T\beta$ to be estimable, we need $\underline{c} = \underline{X}^T\underline{a}$ for some a $\Rightarrow$ $c_1 = \underline{X}_1^T\underline{a}$

$c_2 = \underline{X}_2^T a$

$c_3 = \underline{X}_3^T\underline{a}$

$c_4 = \underline{X}_4^T a$

But $\underline{X}_1 = \underline{X}_2 + \underline{X}_3 + \underline{X}_4$ $\Rightarrow$ c_1 must equal $\underline{X}_2^T a + \underline{X}_3^T\underline{a} + X_4^T\underline{a} = c_2 + c_3 + c_4$

Since $0 = 1 + (-2) + 1$, $(0, 1, -2, 1)\beta = \alpha_1 - 2\alpha_2 + \alpha_3$ is estimable.

In addition, $\underline{c}^T$ has full row rank (1) and $\underline{c}^T$ has fewer rows than rank$(\underline{X}) = 3$ $\Rightarrow$ H_0 is testable.

b. **(8 points)** Suppose the observed response vector is $\mathbf{y} = (1, 0, -1, 0, -1, 1)^T$. Estimate the regression parameter $\boldsymbol{\beta} = (\mu, \alpha_1, \alpha_2, \alpha_3)^T$ under the restriction imposed by H_0 from part (a). Hint: one valid generalized inverse of $\mathbf{X}^T\mathbf{X}$ is

$$\mathbf{G} = \begin{pmatrix} .0938 & .0312 & .0313 & .0313 \\ .0313 & .3437 & -.1562 & -.1562 \\ .0312 & -.1562 & .3438 & -.1563 \\ .0312 & -.1562 & -.1563 & .3438 \end{pmatrix}$$

I wouldn't ask you this question this year (2003). We didn't really talk about this in the non-full rank case, but

$(*) \Rightarrow \hat{\beta} = \hat{\beta}_U - GA^T(AGA^T)^{-1}A\hat{\beta}_U$

One valid choice of $\hat{\beta}_U$ is $\begin{pmatrix} 0 \\ 1/2 \\ -1/2 \\ 0 \end{pmatrix}$ ← mean of y_{11}, y_{12}; ← mean of y_{21}, y_{22}; ← mean of y_{31}, y_{32}

$A = (0, 1, -2, 1) \Rightarrow AG = (.0001, .4999, -1.0001, .5002) \approx (0, 1/2, -1, 1/2)$

$\Rightarrow AGA^T = 1/2 + 2 + 1/2 = 3, \quad A\hat{\beta}_U = \frac{1}{2} + 1 = 1.5$

$$\Rightarrow \hat{\beta} = \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/2 \\ -1 \\ 1/2 \end{pmatrix} \frac{1}{3}\left(\frac{3}{2}\right) = \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/4 \\ -1/2 \\ 1/4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/4 \\ 0 \\ -1/4 \end{pmatrix}$$

formula $(*)$ generalizes formula (8.30) (p. 187 of our text) from the full rank case to the non-full rank case.

c. **(7 points)** Is your estimator from part (b) a least squares estimator of $\boldsymbol{\beta}$ in the original (unrestricted) model? Why, or why not?

No, because $\alpha_1 + \alpha_3 = 2\alpha_2$ corresponds to a real restriction on the model subspace. and therefore the restricted L.S. estimator corresponds to a projection onto a diff't model subspace than does $\hat{\beta}_{unrestricted}$.