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STAT 8260 Exam 2 – Thursday, April 10
SHOW ALL WORK

Name: Answer Key

1. Consider the regression model

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 (x_i^2 - 5)/4 + \varepsilon_i, \quad i = 1, \ldots, 4,$$

where $\varepsilon_1, \ldots, \varepsilon_4$ are i.i.d. with $\mathrm{E}(\varepsilon_i) = 0$, $\mathrm{var}(\varepsilon_i) = \sigma^2$. Suppose $\mathbf{x} = (x_1, \ldots, x_4)^T = (-3, -1, 1, 3)^T$, and $\mathbf{y} = (y_1, \ldots, y_4)^T = (1, 2, 2, 4)^T$.

a. **(6 points)** Find the BLUE of $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)^T$ from this model.

$$\underset{\sim}{X} = \begin{pmatrix} 1 & -3 & 1 \\ 1 & -1 & -1 \\ 1 & +1 & -1 \\ 1 & 3 & 1 \end{pmatrix} \qquad \underset{\sim}{X}^T\underset{\sim}{X} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 3 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & +1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 4\begin{pmatrix} 1 & & \\ & 5 & 0 \\ 0 & \cancel{4} & 1 \end{pmatrix}$$

$$(X^TX)^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad X^Ty = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 3 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \\ 1 \end{pmatrix}$$

$$\hat{\beta} = (X^TX)^{-1}X^Ty = \frac{1}{4}\begin{pmatrix} 1 & & \\ 0 & 1/5 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 9 \\ 9 \\ 1 \end{pmatrix} = \begin{pmatrix} 9/4 \\ 9/20 \\ 1/4 \end{pmatrix}$$

b. **(8 points)** Based on this model, obtain a 95% CI for E(y) when $x = 0$.

$\underline{x}_0^T = (1, 0, -5/4)$

$$\underline{x}_0^T \hat{\beta} = 1(9/4) + 0(9/20) - \frac{5}{4}\left(\frac{1}{4}\right) = \frac{9}{4} - \frac{5}{16} = \frac{36-5}{16} = 31/16$$

$$\cancel{\ } \; MSE = s^2 = \frac{y^Ty - \hat{y}^T\hat{y}}{n-k-1} = (y^Ty - \hat{y}^T\hat{y}) / (\underbrace{4-2-1}_{=1})$$

$$\hat{y} = X\hat{\beta} = \begin{pmatrix} 1 & -3 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 9/4 \\ 9/20 \\ 1/4 \end{pmatrix} = \begin{pmatrix} 9/4 - 27/20 + 1/4 \\ 9/4 - 9/20 - 1/4 \\ 9/4 + 9/20 - 1/4 \\ 9/4 + 27/20 + 1/4 \end{pmatrix} = \begin{pmatrix} \frac{45-27+5}{20} \\ \frac{45-9-5}{20} \\ \frac{45+9-5}{20} \\ \frac{45+27+5}{20} \end{pmatrix}$$

$$= \frac{1}{20}\begin{pmatrix} 23 \\ 31 \\ 49 \\ 77 \end{pmatrix}$$

$\hat{y}^T\hat{y} = \cancel{23.63} \; 24.55$

$y^Ty = 1^2 + 2^2 + 2^2 + 4^2 = 25$

$\Rightarrow MSE = \cancel{25-23.63} = \cancel{1.35}$

$$= \frac{25 - 24.55}{1} = .45$$

$$\underline{x}_0^T (X^TX)^{-1} \underline{x}_0 = \frac{1}{4} \underline{x}_0^T \begin{pmatrix} 1 & & 0 \\ & 1/5 & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -5/4 \end{pmatrix}$$

$$= \frac{1}{4} \underline{x}_0^T \begin{pmatrix} 1 \\ 0 \\ -5/4 \end{pmatrix} = \frac{1}{4}\left(1 + \frac{25}{16}\right) = \frac{1}{4}\left(\frac{41}{16}\right)$$

95% CI is

$$31/16 \pm \underbrace{t_{1-\alpha/2}(\underbrace{n-k-1}_{=1})}_{= 12.71} \sqrt{.45\left(\frac{1}{4}\right)\left(\frac{41}{16}\right)}$$

$$= 31/16 \pm 6.82 = (-4.89, 8.76)$$

2. **(10 points)** Let $y_i = \beta x_i + \varepsilon_i$, $i = 1, 2$, where $\varepsilon_1 \sim N(0, \sigma^2)$ and $\varepsilon_2 \sim N(0, 2\sigma^2)$, and ε_1 and ε_2 are statistically independent. If $x_1 = 1$ and $x_2 = -1$, then obtain the BLUE of β and find the variance of your estimator.

$$\hat{\beta} = (X^T V^{-1} X)^{-1} X^T V^{-1} y \qquad V = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \ V^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$X^T V^{-1} = (1, -1) \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} = (1, -1/2)$$

$$(X^T V^{-1} X)^{-1} = \left[(1, -1/2) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]^{-1} = (3/2)^{-1} = 2/3$$

$$X^T V^{-1} y = (1, -1/2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = y_1 - 1/2\, y_2$$

$$\Rightarrow \hat{\beta} = \frac{2}{3} \left(y_1 - (1/2) y_2 \right)$$

$$Var(\hat{\beta}) = \frac{4}{9} \left(Var(y_1) + \frac{1}{4} Var(y_2) \right)$$

$$= \frac{4}{9} \left(\sigma^2 + \frac{1}{4} 2\sigma^2 \right) = \frac{4}{9} \frac{3}{2} \sigma^2 = \frac{2}{3} \sigma^2$$

3. **(12 points)** In homework # 5, problem # 8.19, you used Lagrange multipliers to show that the least-squares estimator of $\boldsymbol{\beta}$ in the full-rank model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \text{subject to the constraint } \mathbf{C}\boldsymbol{\beta} = \mathbf{0},$$

where $\mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0}$, and $\mathrm{var}(\boldsymbol{\varepsilon}) = \sigma^2\mathbf{I}$, is

$$\hat{\boldsymbol{\beta}}_c = \hat{\boldsymbol{\beta}} - (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{C}^T\{\mathbf{C}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{C}^T\}^{-1}\mathbf{C}\hat{\boldsymbol{\beta}},$$

where $\hat{\boldsymbol{\beta}}$ is the unconstrained least squares estimator of $\boldsymbol{\beta}$. Use a different method (don't use Lagrange multipliers) to prove this result.

Hint: Use the facts that 1) $\boldsymbol{\beta} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\boldsymbol{\mu}$ and 2) subject to the constraint, the model space is $V_0 = C(\mathbf{T})^{\perp} \cap C(\mathbf{X})$, where $\mathbf{T} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{C}^T$.

Under $H_0: C\beta = 0$ $\quad \mu = X\beta \in V_0 = C(T)^{\perp} \cap C(X)$

$\Rightarrow \hat{\beta} = (X^TX)^{-1}X^T\hat{\mu} = (X^TX)^{-1}X^T p(y|V_0)$

$= (X^TX)^{-1}X^T P_{V_0} y \quad$ where $P_{V_0} = P_{C(X)} - P_{C(T)}$

$P_{C(T)} = T(T^TT)^{-1}T^T = X(X^TX)^{-1}C^T[C\cancel{(X^TX)^{-1}X^T}X(X^TX)^{-1}C^T]^{-1}C(X^TX)^{-1}X^T$

$= X(X^TX)^{-1}C^T[C(X^TX)^{-1}C^T]^{-1}C(X^TX)^{-1}X^T$

$P_{C(X)} = X(X^TX)^{-1}X^T \quad \Rightarrow P_{V_0}y = P_{C(X)}y - P_{C(T)}y$

$= X\hat{\beta} - X(X^TX)^{-1}C^T[C(X^TX)^{-1}C^T]^{-1}C\hat{\beta}$

$\cancel{\Rightarrow \hat{\beta}\,(X^TX)^{-1}X^T}$

$\Rightarrow \hat{\beta} = \cancel{(X^TX)^{-1}X^TX}\hat{\beta} - \cancel{(X^TX)^{-1}X^TX}(X^TX)^{-1}C^T[C(X^TX)^{-1}C^T]^{-1}C\hat{\beta}$

$= \hat{\beta} - (X^TX)^{-1}C^T[C(X^TX)^{-1}C^T]^{-1}C\hat{\beta}$

4. Recall the chemical reaction data that you worked on in problem # 8.41 of homework # 5. For these data, y =percent of unchanged starting material, x_1 =temperature, x_2 =concentration of a reagent, and x_3 =time of reaction. I fit the model

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \varepsilon_i, \quad i = 1, \ldots, n = 19,$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)^T \sim N(\mathbf{0}, \sigma^2 \mathbf{I})$. I obtained the following results:

$$\hat{\boldsymbol{\beta}} = \begin{pmatrix} 332.11 \\ -1.55 \\ -1.42 \\ -2.24 \end{pmatrix} \qquad \hat{\text{var}}(\hat{\boldsymbol{\beta}}) = \begin{pmatrix} 349.43 & -1.81 & -1.67 & -.11 \\ -1.81 & .0098 & .0068 & -.0023 \\ -1.67 & .0068 & .022 & -.0094 \\ -.11 & -.0023 & -.0094 & .12 \end{pmatrix}$$

and $s^2 = \text{MSE} = 5.34$, $R^2 = .96$.

a. **(6 points)** Compute the F test statistic for the hypothesis $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$, and give the reference distribution against which this test statistic should be compared for a test of H_0.

$$F = \frac{R^2/k}{(1-R^2)/(n-k-1)} = \frac{.96/3}{.04/(19-3-1)}$$

$$= 120$$

$F(3,15)$

b. **(5 points)** Express the hypothesis $H_0 : 2\beta_1 = \beta_3 = -3$ in the form of the general linear hypothesis.

$H_0 : C\beta = t$ where

$$C = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad t = \begin{pmatrix} -3 \\ -3 \end{pmatrix}$$

$$\beta = (\beta_0, \beta_1, \beta_2, \beta_3)^T$$

c. **(8 points)** Compute the F test statistic for H_0 from part (b) and give the reference distribution for this test statistic.

$$F = \frac{(C\hat{\beta}-t)^T\left[C(X^TX)^{-1}C^T\right]^{-1}(C\hat{\beta}-t)/q}{s^2}$$

$$s^2(X^TX)^{-1} = \mathrm{Var}(\hat{\underline{\beta}}) \Rightarrow (X^TX)^{-1} = \frac{1}{s^2}\begin{pmatrix} 349.43 & -1.81 & & \\ & \ddots & & \\ & & & .12 \end{pmatrix}$$

$$C(X^TX)^{-1}C^T = \frac{1}{s^2}\begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 349.43 & -1.81 & -1.67 & -.11 \\ -1.81 & .0098 & .0068 & -.0023 \\ -1.67 & .0068 & .022 & -.0094 \\ -.11 & -.0023 & -.0094 & .12 \end{pmatrix} \underline{C}^T$$

$$= \frac{1}{s^2}\begin{pmatrix} -3.62 & .0196 & .0136 & -.0046 \\ -.11 & -.0023 & -.0094 & .12 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{s^2}\begin{pmatrix} .0392 & -.0046 \\ -.0046 & .12 \end{pmatrix}$$

$$\left[C(X^TX)^{-1}C^T\right]^{-1} = \frac{s^2}{(.0392)(.12)-(.0046)^2}\begin{pmatrix} .12 & .0046 \\ .0046 & .0392 \end{pmatrix}$$

$$C\hat{\beta}-t = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 332.11 \\ -1.55 \\ -1.42 \\ -2.24 \end{pmatrix} - \begin{pmatrix} -3 \\ -3 \end{pmatrix} = \begin{pmatrix} -3.1 + 3 \\ -2.24 + 3 \end{pmatrix} = \begin{pmatrix} -.1 \\ .76 \end{pmatrix}$$

$$F = (-.1, .76)\begin{pmatrix} .12 & .0046 \\ .0046 & .0392 \end{pmatrix}\begin{pmatrix} -.1 \\ .76 \end{pmatrix} \Big/ 2\left[.0392(.12) - (.0046)^2\right]$$

$$= (-.1, .76)\begin{pmatrix} -.008504 \\ .029332 \end{pmatrix} \Big/ (.004683)2 = 2.47$$

Some rounding error here. True value is 2.57

Ref. dist'n $F(2,15)$