STAT 8260 Exam 1 – Thursday, February 27
SHOW ALL WORK

Name: Answer Key

1. Let

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 6 \\ 0 & 2 & 2 \\ 4 & 0 & 8 \end{pmatrix} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$$

where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are the columns of $\mathbf{A}$. Let $V = \mathcal{L}(\mathbf{a}_2, \mathbf{a}_3)$ and suppose $\mathbf{y} \sim N_3(\boldsymbol{\mu}, \sigma^2\mathbf{I}_3)$.

a. **(9 points)** Find $\mathbf{P}_V$ and $\mathbf{P}_{C(\mathbf{A})}$.

$V = \mathcal{L}(\underline{a}_2, \underline{a}_3) = \mathcal{L}(\underline{a}_1, \underline{a}_2) = C(\underline{A})$ since $\underline{a}_3 = 2\underline{a}_1 + \underline{a}_2$

$P_V = P_{C(A)} = \underline{O}\,\underline{O}^T$ where $\underline{O} = \left(\frac{1}{\|\underline{a}_1\|}\underline{a}_1, \frac{1}{\|\underline{a}_2\|}\underline{a}_2\right) = \begin{pmatrix} 3/5 & 0 \\ 0 & 1 \\ 4/5 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 3/5 & 0 \\ 0 & 1 \\ 4/5 & 0 \end{pmatrix}\begin{pmatrix} 3/5 & 0 & 4/5 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 9/25 & 0 & 12/25 \\ 0 & 1 & 0 \\ 12/25 & 0 & 16/25 \end{pmatrix}$$

b. **(7 points)** Find the eigenvalues (and their multiplicities) of $\mathbf{P}_V$.

All eigenvalues are 0's an 1's of any projection matrix w/ multiplicity of 1 equal to dim(V).

So the eigenvalues are 1, w/ multiplicity dim(V) = 2 and 0, w/ multiplicity 1

c. **(7 points)** Are $\mathbf{y}^T\mathbf{P}_V\mathbf{y}$ and $\mathbf{y} - \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-}\mathbf{A}^T\mathbf{y}$ independent? Why or why not?

$$\underline{y} - \underline{A}(\underline{A}^T\underline{A})^{-}\underline{A}^T\underline{y} = (\underline{I} - \underbrace{\underline{A}(\underline{A}^T\underline{A})^{-}\underline{A}^T}_{=\underline{P}_V})\underline{y}$$

$$= \underline{P}_{V^\perp}\underline{y}$$

Since $\underline{P}_{V^\perp}\underline{P}_V = \underline{0}$, they are independant.

d. **(7 points)** What is the distribution of $\frac{1}{\sigma^2}||\mathbf{y}-\mathbf{c}||^2$, where $\mathbf{c}$ is a 3×1 vector of constants?

$$\frac{1}{\sigma^2}||\underline{y}-\underline{c}||^2 = \frac{1}{\sigma^2}\underbrace{(\underline{y}-\underline{c})^T\underline{I}(\underline{y}-\underline{c})}_{=Q}$$

$$\underline{y}-\underline{c} \sim N_3(\underline{\mu}-\underline{c}, \cancel{\sigma^2}\sigma^2\underline{I})$$

Since $\underline{I}$, the matrix in the quadratic form Q is idempotent w/ rank 3

$$\Rightarrow \frac{1}{\sigma^2}||\underline{y}-\underline{c}||^2 \sim \chi^2\left(3, \frac{1}{2\sigma^2}(\underline{\mu}-\underline{c})^T(\underline{\mu}-\underline{c})\right)$$

$$= \chi^2\left(3, \frac{1}{2\sigma^2}||\underline{\mu}-\underline{c}||^2\right)$$

e. **(7 points)** Now suppose $\boldsymbol{\mu} = (1/3, 0, -1/4)^T$ and find $\mathrm{E}(\mathbf{y}^T \mathbf{P}_V \mathbf{y})$.

$E(\underline{y}^T \underline{P}_V \underline{y}) = tr(\underline{P}_V \sigma^2 \underline{I}) + \underline{\mu}^T \underline{P}_V \underline{\mu}$

$= \sigma^2 \underbrace{tr(\underline{P}_V)}_{=\dim(V)=2} + \|\underline{P}_V \underline{\mu}\|^2$. However $\underline{\mu}$ is in $V^\perp$ (it is orthogonal to both $\underline{a}_1$ and $\underline{a}_2$ where $\underline{a}_1, \underline{a}_2$ are a basis for V)

$\Rightarrow \underline{P}_V \underline{\mu} = \underline{0}$

$\Rightarrow E(\underline{y}^T \underline{P}_V \underline{y}) = 2\sigma^2$

2. Suppose $\mathbf{y} \sim N_3(\boldsymbol{\mu}, \Sigma)$ where $\boldsymbol{\mu} = (2, 5, -2)^T$ and

$$\Sigma = \begin{pmatrix} 9 & 0 & 3 \\ 0 & 1 & -1 \\ 3 & -1 & 6 \end{pmatrix}.$$

a. **(6 points)** Which pairs of variables in $\mathbf{y}$ are independent?

y_1, y_2 are independent

b. ($\cancel{13}$ 14 **points**) Find the distribution of $\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} | y_1$.

$$E\left(\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \Big| y_1\right) = E\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} + \operatorname{cov}\left(\begin{pmatrix} y_2 \\ y_3 \end{pmatrix}, y_1\right) \operatorname{var}(y_1)^{-1} (y_1 - E(y_1))$$

$$= \begin{pmatrix} 5 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} \frac{1}{9} (y_1 - 2) = \begin{pmatrix} 5 \\ -2 + \frac{1}{3} y_1 - \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ -\frac{8}{3} + \frac{1}{3} y_1 \end{pmatrix}$$

$$\operatorname{Var}\left(\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \Big| y_1\right) = \operatorname{var}\left(\begin{pmatrix} y_2 \\ y_3 \end{pmatrix}\right) - \operatorname{Cov}\left(\begin{pmatrix} y_2 \\ y_3 \end{pmatrix}, y_1\right) \operatorname{Var}(y_1)^{-1} \operatorname{Cov}\left(y_1, \begin{pmatrix} y_2 \\ y_3 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 1 & -1 \\ -1 & 6 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} \frac{1}{9} (0, 3) = \begin{pmatrix} 1 & -1 \\ -1 & 6 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ -1 & 5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \Big| y_1 \sim N_2\left(\begin{pmatrix} 5 \\ -\frac{8}{3} + \frac{1}{3} y_1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 5 \end{pmatrix}\right)$$

c. ($\cancel{8}$ 7 **points**) Find the partial correlation coefficient $\rho_{23|1}$ ($\rho_{23 \cdot 1}$ in our book's notation).

$= \operatorname{corr}(y_2, y_3)$ computed from $\operatorname{Var}\left(\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \Big| y_1\right)$

$$= \begin{pmatrix} 1 & -1 \\ -1 & 5 \end{pmatrix}$$

$$\Rightarrow \rho_{23|1} = \frac{-1}{\sqrt{1(5)}} = -\frac{1}{\sqrt{5}}$$

3. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be independent random samples from $N(\mu_1, \sigma^2)$ and $N(\mu_2, 2\sigma^2)$, respectively. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ and $\bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$..

a. **(8 points)** Find the distribution of

$$2\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sum_{i=1}^n (Y_i - \bar{Y})^2}.$$

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\bar{X})^2 \sim \chi^2(n-1), \quad \frac{1}{2\sigma^2}\sum(Y_i-\bar{Y})^2 \sim \chi^2(n-1)$$

independent

So $$\frac{\frac{1}{\sigma^2}\sum(X_i-\bar{X})^2/(n-1)}{\frac{1}{2\sigma^2}\sum(Y_i-\bar{Y})^2/(n-1)} = 2\,\frac{\sum(X_i-\bar{X})^2/(n-1)}{\sum(Y_i-\bar{Y})^2/(n-1)} \sim F(n-1, n-1)$$

b. **(8 points)** Find the mean and variance of

$$\underbrace{\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \bar{X})^2}_{\sim \chi^2(n-1)} + \underbrace{\frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - \bar{Y})^2}_{\sim \chi^2(n-1)}.$$

independent

$\Rightarrow$ Sum is $\chi^2(n-1+n-1) = \chi^2(2n-2)$

Mean of $\chi^2(df)$ is df, and variance is $2(df)$

$\Rightarrow$ Mean here is $2n-2$

and variance is $4n-4$

4. Let $\mathbf{x} = (x_1, x_2)^T$, $\mathbf{c} = (1, 1, 1)^T$,

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 5 & -3 \end{pmatrix}, \quad \text{and} \quad \mathbf{B} = \frac{1}{7}\begin{pmatrix} 1 & 2 & 0 \\ -3 & 1 & 0 \end{pmatrix}.$$

7

a. (**8 points)** Show that **B** is a generalized inverse of **A**.

$$ABA = \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 5 & -3 \end{pmatrix} \frac{1}{7}\begin{pmatrix} 1 & 2 & 0 \\ -3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 5 & -3 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = \underline{A}\,\underline{I} = \underline{A} \checkmark$$

7

b. (**6 points)** Show that $\mathbf{Ax} = \mathbf{c}$ is not a consistent system of equations.

$\underline{A}\,\underline{x} = \underline{c}$ consistent iff $AA^{-}c = c$ for any generalized inverse A^{-} of A.

So check if $ABc = c$

$$ABc = \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 5 & -3 \end{pmatrix} \frac{1}{7}\begin{pmatrix} 1 & 2 & 0 \\ -3 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 5 & -3 \end{pmatrix} \frac{1}{7}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 7 \\ 7 \\ 21 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \neq \underline{c}$$

$\Rightarrow \underline{A}\,\underline{x} = \underline{c}$ is not consistent

c. **(6 points)** Change $\mathbf{c}$ so that the system $\mathbf{Ax} = \mathbf{c}$ is consistent.

$$\underline{A} = \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \end{pmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \end{matrix} \text{ rows of } \underline{A}$$

Notice $\underline{a}_3 = 2\underline{a}_1 + \underline{a}_2$

So for $\underline{A}\underline{x} = \underline{c}$ to be consistent the same relationship as holds in $\underline{A}$ must hold in $\underline{c}$ that is, must have $c_3 = 2c_1 + c_2$

So $\underline{c} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ will work