STAT 8260 Exam 1 – Thursday, February 27
SHOW ALL WORK

Name:______________________________________

1. Let

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 6 \\ 0 & 2 & 2 \\ 4 & 0 & 8 \end{pmatrix} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$$

where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are the columns of $\mathbf{A}$. Let $V = \mathcal{L}(\mathbf{a}_2, \mathbf{a}_3)$ and suppose $\mathbf{y} \sim N_3(\boldsymbol{\mu}, \sigma^2 \mathbf{I}_3)$.

a. **(9 points)** Find $\mathbf{P}_V$ and $\mathbf{P}_{C(\mathbf{A})}$.

b. **(7 points)** Find the eigenvalues (and their multiplicities) of $\mathbf{P}_V$.

c. **(7 points)** Are $\mathbf{y}^T \mathbf{P}_V \mathbf{y}$ and $\mathbf{y} - \mathbf{A}(\mathbf{A}^T\mathbf{A})^- \mathbf{A}^T \mathbf{y}$ independent? Why or why not?

d. **(7 points)** What is the distribution of $\frac{1}{\sigma^2}||\mathbf{y} - \mathbf{c}||^2$, where $\mathbf{c}$ is a 3×1 vector of constants?

e. **(7 points)** Now suppose $\boldsymbol{\mu} = (1/3, 0, -1/4)^T$ and find $\mathrm{E}(\mathbf{y}^T \mathbf{P}_V \mathbf{y})$.

2. Suppose $\mathbf{y} \sim N_3(\boldsymbol{\mu}, \Sigma)$ where $\boldsymbol{\mu} = (2, 5, -2)^T$ and

$$\Sigma = \begin{pmatrix} 9 & 0 & 3 \\ 0 & 1 & -1 \\ 3 & -1 & 6 \end{pmatrix}.$$

a. **(6 points)** Which pairs of variables in $\mathbf{y}$ are independent?

b. **(14 points)** Find the distribution of $\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} | y_1$.

c. **(7 points)** Find the partial correlation coefficient $\rho_{23|1}$ ($\rho_{23\cdot 1}$ in our book's notation).

3. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be independent random samples from $N(\mu_1, \sigma^2)$ and $N(\mu_2, 2\sigma^2)$, respectively. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ and $\bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$.

   a. **(8 points)** Find the distribution of

$$2\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sum_{i=1}^n (Y_i - \bar{Y})^2}.$$

   b. **(8 points)** Find the mean and variance of

$$\frac{1}{\sigma^2}\sum_{i=1}^n (X_i - \bar{X})^2 + \frac{1}{2\sigma^2}\sum_{i=1}^n (Y_i - \bar{Y})^2.$$

4. Let $\mathbf{x} = (x_1, x_2)^T$, $\mathbf{c} = (1, 1, 1)^T$,

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 5 & -3 \end{pmatrix}, \quad \text{and} \quad \mathbf{B} = \frac{1}{7}\begin{pmatrix} 1 & 2 & 0 \\ -3 & 1 & 0 \end{pmatrix}.$$

a. **(7 points)** Show that $\mathbf{B}$ is a generalized inverse of $\mathbf{A}$.

b. **(7 points)** Show that $\mathbf{Ax} = \mathbf{c}$ is not a consistent system of equations.

c. **(6 points)** Change $\mathbf{c}$ so that the system $\mathbf{Ax} = \mathbf{c}$ is consistent.