STAT 8260 Homework #3 Solution

Spring '08

1. $E(\underline{x}) = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$, $E(\underline{y}) = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$, $var(\underline{x}) = \begin{pmatrix} 4 & 2 & 3 \\ 2 & 9 & 1 \\ 3 & 1 & 5 \end{pmatrix}$, $var(\underline{y}) = \begin{pmatrix} 4 & -2 & 3 \\ -2 & 6 & 2 \\ 3 & 2 & 8 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \end{pmatrix}$, $\underline{B} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$, $\underline{x}, \underline{y}$ indep.

a) $\underline{x}, \underline{y}$ indep $\Rightarrow$ $cov(\underline{x}, \underline{y}) = \underline{0}$

b.) $var(x+y) = var(x) + var(y) + \underbrace{cov(x,y) + cov(y,x)}_{=0}$

$= \begin{pmatrix} 8 & 0 & 6 \\ 0 & 15 & 3 \\ 6 & 3 & 13 \end{pmatrix}$

c) $cov(Ax, Bx) = A \underbrace{cov(\underline{x}, \underline{x})}_{= var(x)} B^T = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 3 \\ 2 & 9 & 1 \\ 3 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 10 & 4 & 13 \\ 9 & 28 & 8 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 17 & 9 \\ 1 & 36 & -20 \end{pmatrix}$

d.) $E(Ax + Ay) = A(E(x) + E(y)) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \end{pmatrix} \left[\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \right] = \begin{pmatrix} 24 \\ 31 \end{pmatrix}$

e) $var(Ax) = A \, var(x) A^T = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 3 \\ 2 & 9 & 1 \\ 3 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 36 & 25 \\ 25 & 92 \end{pmatrix}$

f.) $corr(Ax, x) = V_{Ax}^{-1/2} \, cov(Ax, x) \, V_x^{-1/2} = V_{Ax}^{-1/2} A \, var(x) V_x^{-1/2}$

$= \begin{pmatrix} 1/\sqrt{36} & 0 \\ 0 & 1/\sqrt{92} \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 3 \\ 2 & 9 & 1 \\ 3 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/\sqrt{5} \end{pmatrix}$

$= \begin{pmatrix} .833 & .222 & .969 \\ .469 & .973 & .373 \end{pmatrix}$

g. $\mathrm{corr}(Ax, Bx) = V_{Ax}^{-1/2}\ \mathrm{cov}(Ax, Bx)\ V_{Bx}^{-1/2}$

$= \left[\mathrm{diag}\left(\mathrm{diag}\left(A\,\mathrm{var}(x)\,A^T\right)\right)\right]^{-1/2} A\ \mathrm{var}(x)\,B^T\left[\mathrm{diag}\left(\mathrm{diag}\left(B\,\mathrm{var}(x)\,B^T\right)\right)\right]^{-1/2}$

$= \begin{pmatrix} -.289 & .708 & .433 \\ .0602 & .938 & -.602 \end{pmatrix}$

2.

2. Suppose $X = \begin{cases} 1 & \text{w/ prob } p_X \\ 0 & \text{w/ prob } 1-p_X \end{cases}$, $Y = \begin{cases} 1 & \text{w/ prob } p_Y \\ 0 & \text{w/ prob } 1-p_Y \end{cases}$

Suppose $0 = \operatorname{cov}(X,Y) \Rightarrow 0 = E(XY) - E(X)E(Y) \Rightarrow E(XY) = p_X p_Y$

The random variable $Z = XY$ takes values 0 and 1 with $Pr(Z=1) = E(Z) = E(XY) = p_X p_Y$

$\Rightarrow Pr(X=1, Y=1) = Pr(Z=1) = p_X p_Y$

and
$$Pr(X=1, Y=0) = Pr(X=1) - Pr(X=1, Y=1) = p_X - p_X p_Y = p_X(1-p_Y)$$
$$Pr(X=0, Y=1) = Pr(Y=1) - Pr(X=1, Y=1) = p_Y - p_X p_Y = p_Y(1-p_X)$$
$$Pr(X=0, Y=0) = 1 - p_X p_Y - p_X(1-p_Y) - p_Y(1-p_X) = (1-p_X)(1-p_Y)$$

$\Rightarrow X, Y$ are independent.

2. (continued) Now the general case. Suppose U takes on two values, u_1, u_2 with probabilities p_x, $1-p_x$, respectively, and W takes values w_1, w_2 with probabilities p_y, $1-p_y$. Then U and W can be written

$$U = Xu_1 + (1-X)u_2 = u_2 + X(u_1 - u_2)$$
$$W = Yw_1 + (1-Y)w_2 = w_2 + Y(w_1 - w_2)$$

So $\operatorname{Cov}(U,W) = 0 \Rightarrow \operatorname{cov}\big(u_2 + X(u_1 - u_2), w_2 + Y(w_1 - w_2)\big) = 0$

$$\Rightarrow \underbrace{(u_1 - u_2)(w_1 - w_2)}_{\text{both nonzero}} \operatorname{cov}(X,Y) = 0 \Rightarrow \operatorname{cov}(X,Y) = 0 \Rightarrow X, Y \text{ independent}$$

But if X, Y are independent so are U, W because U is a function of X only and W a function of Y only.

3. $\underline{y} = \begin{pmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{22} \\ y_{23} \end{pmatrix}$ $\quad y_{ij} = G_i + e_{ij}$ or $\underline{y} = \underline{X}\,\underline{g} + \underline{e}$

where $\underline{X} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$, $\underline{g} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$, $\underline{e} = (e_{11}, \ldots, e_{23})^T$

~~$\underline{g} \sim (\mu \underline{j}_2, \sigma_G^2 I_2)$ indep. of $\underline{e} \sim N$~~

$E(\underline{g}) = \mu \underline{j}_2$, $\operatorname{var}(\underline{g}) = \sigma_G^2 I_2$, $E(\underline{e}) = \underline{0}$, $\operatorname{var}(\underline{e}) = \sigma^2 I$

$\operatorname{cov}(\underline{g}, \underline{e}) = 0$

indep.

3. $\mathrm{var}(\underline{y}) = \mathrm{var}(\underline{X}\,\underline{g} + \underline{e}) = \underline{X}\,\mathrm{var}(\underline{g})\,\underline{X}^T + \mathrm{var}(\underline{e})$

$$= \sigma_G^2 \underline{X}\,\underline{X}^T + \sigma^2 \underline{I}_n = \sigma_G^2 \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} + \sigma^2 \underline{I}_n$$

$$= \begin{pmatrix} \sigma_G^2 \underline{J}_{3,3} + \sigma^2 \underline{I}_3 & \underline{0} \\ \underline{0} & \sigma_G^2 \underline{J}_{3,3} + \sigma^2 \underline{I}_3 \end{pmatrix}$$

$$\mathrm{corr}(\underline{y}) = \mathrm{diag}\left(\frac{1}{\sqrt{\sigma_G^2+\sigma^2}}\,\underline{j}_6\right) \mathrm{var}(\underline{y})\, \mathrm{diag}\left(\frac{1}{\sqrt{\sigma_G^2+\sigma^2}}\,\underline{j}_6\right)$$

$$= \frac{1}{\sigma_G^2+\sigma^2}\,\underline{I}_6\,\mathrm{var}(\underline{y})\,\underline{I}_6 = \begin{pmatrix} \frac{\sigma_G^2}{\sigma_G^2+\sigma^2} J_{3,3} + \frac{\sigma^2}{\sigma_G^2+\sigma^2} I_3 & \underline{0} \\ \underline{0} & \frac{\sigma_G^2}{\sigma_G^2+\sigma^2} J_{3,3} + \frac{\sigma^2}{\sigma_G^2+\sigma^2} I_3 \end{pmatrix}$$

$$= \begin{pmatrix} \rho J_{3,3} + (1-\rho) I_3 & \underline{0} \\ \underline{0} & \rho J_{3,3} + (1-\rho) I_3 \end{pmatrix} \quad \text{where } \rho = \frac{\sigma_G^2}{\sigma_G^2+\sigma^2}$$

4. $E(\underline{x}) = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \equiv \underline{\mu}_{\underline{x}}$, $\mathrm{var}(\underline{x}) = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix} \equiv \Sigma_{\underline{x}}$

$Q(\underline{x}) = \sum_{i=1}^{3} (x_i - \bar{x})^2 = \| x - \bar{x} \underline{j}_3 \|^2 = \| p(x | \mathcal{L}(\underline{j}_3)^\perp) \|^2$

$= \| P_{\mathcal{L}(\underline{j}_3)^\perp} \underline{x} \|^2 = \underline{x}^T P_{\mathcal{L}(\underline{j}_3)^\perp} \underline{x} = \underline{x}^T (\underline{I} - \frac{1}{n} J_{n,n}) \underline{x} \quad (n=3)$

Using the general result $E(x^T A x) = tr(A \Sigma_x) + \mu_x^T A \mu_x$

we have $E(Q(x)) = tr\left[(I - \frac{1}{3} J_{3,3}) \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix} \right]$

$+ (2, 3, 4)(I - \frac{1}{3} J_{3,3}) \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = tr \begin{pmatrix} 5/3 & -2/3 & -2 \\ -1/3 & 1/3 & 0 \\ -4/3 & 1/3 & 2 \end{pmatrix} + 2 = 6$

5 a) $var(\bar{X}) = var\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n^2}\sum var(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2$

or

$$var(\bar{X}) = var\left(\frac{1}{n}\underline{j}_n^T \underline{X}\right) = \frac{1}{n^2}\underline{j}_n^T var(\underline{X})\underline{j}_n = \frac{1}{n^2}\underline{j}_n^T \begin{pmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_n^2 \end{pmatrix} \underline{j}_n$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2$$

b.) $E(Q) = k_n E\left[\sum (x_i - \bar{x})^2\right] = tr\left[\Sigma - \frac{1}{n}J_{n,n}\Sigma\right] + \underline{\mu}^T\left[I_n - \frac{1}{n}J_{n,n}\right]\underline{\mu}$

$$= \left\{ tr\left[\begin{pmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{pmatrix} - \frac{1}{n}\underline{j}_n(\sigma_1^2, \ldots, \sigma_n^2)\right] + \underline{\mu}^T\underline{\mu} - \frac{1}{n}\left(\sum \mu_i\right)\left(\sum \mu_i\right)\right\} k_n$$

$$= \left\{\sum_i\left(\sigma_i^2 - \frac{1}{n}\sigma_i^2\right) + n\mu^2 - n\mu^2\right\} k_n = \left[\sum_i \sigma_i^2\left(\frac{n-1}{n}\right)\right] k_n = k_n \frac{n-1}{n}\sum_i \sigma_i^2 = var(\bar{X})$$

if $k_n = \frac{1}{n(n-1)}$

#6. a) $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \Rightarrow |\Sigma| = \sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2 = \sigma_1^2\sigma_2^2(1-\rho^2)$

$\Rightarrow \Sigma^{-1} = \dfrac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}^T \Sigma^{-1} \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix} = \dfrac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \left[\sigma_2^2(x_1-\mu_1)^2 + \sigma_1^2(x_2-\mu_2)^2 - 2\rho\sigma_1\sigma_2(x_1-\mu_1)(x_2-\mu_2)\right]$

$= \dfrac{1}{1-\rho^2}\left[\left(\dfrac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\dfrac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\dfrac{x_1-\mu_1}{\sigma_1}\right)\left(\dfrac{x_2-\mu_2}{\sigma_2}\right)\right] = \dfrac{1}{1-\rho^2} Q(\underline{x})$

$\Rightarrow f_X(\underline{x}) = (2\pi)^{-n/2} \det(\Sigma)^{-1/2} \exp\{-(\underline{x}-\underline{\mu})^T \Sigma (\underline{x}-\underline{\mu})/2\}$

$= (2\pi)^{-2/2}\left(\sigma_1^2\sigma_2^2(1-\rho^2)\right)^{-1/2} \exp\{-Q(\underline{x})/[2(1-\rho^2)]\}$

$= \left[(2\pi)^2\sigma_1^2\sigma_2^2(1-\rho^2)\right]^{-1/2} \exp\{-\tfrac{1}{2}Q(\underline{x})/(1-\rho^2)\}$

b.) By Thm 4.4D (or the Thm on p. $\boxed{74}$ of our notes) the conditional distribution of $X_2 | X_1 = x_1$ is

$N_1\left(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1),\ \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{21}^T\right)$

Where the cond'l mean is $\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1-\mu_1) = \mu_2 + \rho\sigma_1\sigma_2\dfrac{1}{\sigma_1^2}(x_1-\mu_1)$

$= \mu_2 + \rho\sigma_2\left(\dfrac{x_1-\mu_1}{\sigma_1}\right)$

and the cond'l variance is

$\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{21}^T = \sigma_2^2 - \rho\sigma_1\sigma_2\dfrac{1}{\sigma_1^2}\rho\sigma_1\sigma_2$

$= \sigma_2^2(1-\rho^2)$

 # 7. $\underset{\sim}{y} \sim N_3(\underset{\sim}{\mu}, \Sigma)$ where $\underset{\sim}{\mu} = (2,-1,3)^T$, $\Sigma = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

a) find dist'n of $z = 4y_1 - 6y_2 + y_3 = \begin{pmatrix} 4 \\ -6 \\ 1 \end{pmatrix}^T \underset{\sim}{y}$

$$\sim N\Bigg(\underbrace{\begin{pmatrix} 4 \\ -6 \\ 1 \end{pmatrix}^T \underset{\sim}{\mu}}_{= 4(2) - 6(-1) + 1(3) = 17},\ \underbrace{\begin{pmatrix} 4 \\ -6 \\ 1 \end{pmatrix}^T \Sigma \begin{pmatrix} 4 \\ -6 \\ 1 \end{pmatrix}}\Bigg) = (4\ \ -6\ \ 1)\begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ -6 \\ 1 \end{pmatrix}$$

$$= (10\ \ -7\ \ -3)\begin{pmatrix} 4 \\ -6 \\ 1 \end{pmatrix} = 79$$

b) $\underset{\sim}{z} = \begin{pmatrix} y_1 - y_2 + y_3 \\ 2y_1 + y_2 - y_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \underset{\sim}{y} \sim N_2(\underset{\sim}{\mu}^*, \Sigma^*)$

where $\underset{\sim}{\mu}^* = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \underset{\sim}{\mu} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$

$$\Sigma^* = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \Sigma \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}^T = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 2 \\ 9 & 3 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 23 \end{pmatrix}$$

c) $f(y_2 \mid y_1, y_3)$ rewrite y as $\underset{\sim}{y} = \begin{pmatrix} y_1 \\ y_3 \\ y_2 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}\right)$

then $y_2 \mid y_1, y_3 \sim N_1(\tilde{\mu}_2, \tilde{\sigma}_2^2)$ where

$$\tilde{\sigma}_2^2 = 2 - (1,1)\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 - (1,1)\begin{pmatrix} 1/4 & 0 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 - \frac{1}{4} - \frac{1}{3}$$

$$= 17/12$$

and

$$\tilde{\mu}_2 = -1 + (1,1)\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}^{-1}\begin{pmatrix} y_1 - 2 \\ y_3 - 3 \end{pmatrix} = -1 + \left(\frac{1}{4}, \frac{1}{3}\right)\begin{pmatrix} y_1 - 2 \\ y_3 - 3 \end{pmatrix}$$

$$= -1 + \frac{1}{4}y_1 - \frac{1}{2} + \frac{1}{3}y_3 - 1 = -\frac{5}{2} + \frac{1}{4}y_1 + \frac{1}{3}y_3$$

#7. d) Again rewriting (re-ordering) $\underline{y}$ facilitates applying the Theorem on p.73 of our notes.

let $\underline{y} = \begin{pmatrix} y_3 \\ y_1 \\ y_2 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 1 \\ 0 & 4 & 1 \\ 1 & 1 & 2 \end{pmatrix} \right)$

$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Big| y_3 \sim N_2(\underline{\gamma}, \underline{\Delta})$

where $\underline{\gamma} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (3)^{-1} (y_3 - 3) = \begin{pmatrix} 2 \\ -1 + \frac{1}{3} y_3 - 1 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ -2 + \frac{1}{3} y_3 \end{pmatrix}$

and

$\underline{\Delta} = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{1}{3} (0, 1) = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & +1/3 \end{pmatrix}$

$= \begin{pmatrix} 4 & 1 \\ 1 & 5/3 \end{pmatrix}$

e) $\rho_{12} = \dfrac{\text{cov}(y_1, y_2)}{\sqrt{\text{var}(y_1)\,\text{var}(y_2)}} = \dfrac{1}{\sqrt{(4)(2)}} = \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}$

$= .3536$

$\rho_{12.3}$ is $\text{corr}(y_1, y_2)$ computed from $\underline{\Delta}$ above

$\Rightarrow \rho_{12.3} = \dfrac{1}{\sqrt{4(5/3)}} = \dfrac{1}{2\sqrt{5/3}} = \dfrac{\sqrt{3}}{2\sqrt{5}} = .3873$

8. $\underline{w} \equiv \begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2\left(\underbrace{\begin{pmatrix} 66 \\ 70 \end{pmatrix}}_{\underset{\mu}{\parallel}}, \underbrace{\begin{pmatrix} 2.5^2 & (.3)(2.5)(2.7) \\ \text{symm} & 2.7^2 \end{pmatrix}}_{\equiv \Sigma} \right)$

Let $U = Y - X = \underline{a}^T \underline{w}$ where $\underline{a} = (-1, 1)^T$

$\Rightarrow U \sim N(\underline{a}^T \underline{\mu}, \underline{a}^T \Sigma \underline{a})$ where $\underline{a}^T \underline{\mu} = 4$, $\underline{a}^T \Sigma \underline{a} = 9.49$

$$Pr(U \geq 0) = Pr\left(\frac{U-4}{\sqrt{9.49}} \geq \frac{0-4}{\sqrt{9.49}} \right) = Pr(\underset{\sim N(0,1)}{Z} \geq -1.30)$$

$$= .903$$

As ρ increases, var(U) decreases, so the probability increases.

9.

#9. a.) $X_1, \dots, X_{n_1} \overset{iid}{\sim} N(\mu_1, \sigma_1^2)$, $Y_1, \dots, Y_{n_2} \overset{iid}{\sim} N(\mu_2, \sigma_2^2)$

$\frac{1}{\sigma_1^2} \sum_{i=1}^{n_1} (X_i - \mu_1)^2 = \sum_{i=1}^{n_1} Z_i^2$ where $Z_i = \frac{X_i - \mu_1}{\sigma_1}$

$Z_1, \dots, Z_{n_1} \overset{iid}{\sim} N(0,1)$, so $\frac{1}{\sigma_1^2} \sum_{i=1}^{n_1} (X_i - \mu_1)^2 \sim \chi^2(n_1)$

$\frac{1}{\sigma_1^2} \sum_{i=1}^{n_1} (X_i - c_i)^2 = \sum_{i=1}^{n_1} W_i^2$ where $W_i = \frac{X_i - c_i}{\sigma_1}$

$W_1, \dots, W_n$ are i.i.d. normal w/ means $\frac{\mu_1 - c_1}{\sigma_1}, \dots, \frac{\mu_1 - c_{n_1}}{\sigma_1}$

and common variance 1 $\Rightarrow \frac{1}{\sigma_1^2} \sum_{i=1}^{n_1} (X_i - c_i)^2 \sim \chi^2(n_1, \lambda)$

where $\lambda = \frac{1}{2\sigma_1^2} \sum (\mu_1 - c_i)^2$

9. b.) $\frac{n_1}{\sigma_1^2}(\bar{X}-\mu_0)^2$ where μ_0 is an arbitrary constant

$$\bar{X} \sim N\left(\mu_1, \frac{\sigma_1^2}{n_1}\right) \Rightarrow \bar{X}-\mu_0 \sim N\left(\mu_1-\mu_0, \frac{\sigma_1^2}{n_1}\right)$$

$$\Rightarrow \frac{\bar{X}-\mu_0}{\sigma_1/\sqrt{n_1}} \sim N\left(\frac{\mu_1-\mu_0}{\sigma_1/\sqrt{n_1}}, 1\right)$$

$$\Rightarrow \left(\frac{\bar{X}-\mu_0}{\sigma_1/\sqrt{n_1}}\right)^2 = \frac{n_1}{\sigma_1^2}(\bar{X}-\mu_0)^2 \sim \chi^2\left(1, \frac{(\mu_1-\mu_0)^2 n_1}{2\sigma_1^2}\right)$$

$$\Rightarrow \frac{n_1}{\sigma_1^2}(\bar{X}-\mu_1)^2 \sim \chi^2\left(1, \frac{(\mu_1-\mu_1)^2 n_1}{2\sigma_1^2}\right) = \chi^2(1,0) = \chi^2(1)$$

c) $\frac{n_1}{\sigma_1^2}(\bar{X}-\mu_1)^2 + \frac{n_2}{\sigma_2^2}(\bar{Y}-\mu_2)^2$. From part (b) we know that each of the two terms are $\chi^2(1)$

Since the two terms are independent, their sum is $\chi^2(1+1) = \chi^2(2)$

d.) $$\frac{n_2-1}{n_1}\frac{\Sigma(X_i-\mu_0)^2}{\Sigma(Y_i-\bar{Y})^2} = \frac{\left[\frac{1}{\sigma^2}\Sigma(X_i-\mu_0)^2\right]/n_1}{\left[\frac{1}{\sigma^2}\Sigma(Y_i-\bar{Y})^2\right]/(n_2-1)}$$

by part (a), $\frac{1}{\sigma^2}\Sigma(X_i-\mu_0)^2 \sim \chi^2(n_1, \lambda)$ where

$$\lambda = \frac{n_1}{2\sigma^2}(\mu_1-\mu_0)^2 = \frac{n_1}{2\sigma^2}(\mu_1-\mu_0)^2$$

and by example 5.5 (see also the Thm on p. ~~93~~ 94 of the notes, part 2)

$$\frac{1}{\sigma^2}\Sigma(Y_i-\bar{Y})^2 \sim \chi^2(n_2-1) \Rightarrow \frac{n_2-1}{n_1}\frac{\Sigma(X_i-\mu_0)^2}{\Sigma(Y_i-\bar{Y})^2} \sim F(n_1, n_2-1, \lambda)$$

Where $\lambda = \frac{n_1}{2\sigma^2}(\mu_1-\mu_0)^2$

10. Let $D_i = X_i - Y_i = (1, -1)\,\underline{w}_i$ where $\underline{w}_i = \begin{pmatrix} X_i \\ Y_i \end{pmatrix}$

$$\Rightarrow D_1, \dots, D_n \overset{iid}{\sim} N\Big(\mu_1 - \mu_2,\ \underbrace{(1,-1)\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{=\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\Big)$$

$$\Rightarrow \bar{D} - \delta \sim N\left(\mu_1 - \mu_2 - \delta,\ \frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{n}\right)$$

$$\Rightarrow K_1(\bar{D} - \delta) \sim N\left(K_1[\mu_1 - \mu_2 - \delta],\ 1\right)$$

where $K_1 = \dfrac{\sqrt{n}}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}}$

In addition, $\dfrac{1}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \sum (D_i - \bar{D})^2 \sim \chi^2(n-1)$

(see Example 5.5 or Thm on p. ~~93~~ 94 of the notes, part 2)

Therefore, $\dfrac{(\bar{D} - \delta)K_1}{\left\{K_2 \sum (D_i - \bar{D})^2\right\}^{1/2}} \sim t(m, \theta)$

where $K_2 = \dfrac{1}{(n-1)(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)}$ and $m = n-1$, $\theta = (\mu_1 - \mu_2 - \delta)K_1$

$$\Rightarrow T = K \frac{(\bar{X} - \bar{Y}) - \delta}{\left\{\sum\left[(X_i - Y_i) - (\bar{X} - \bar{Y})\right]^2\right\}^{1/2}} \sim t(m, \theta)$$

where $K = \dfrac{K_1}{\sqrt{K_2}} = \sqrt{n(n-1)}$

11. (5.23 in text) $\underline{y} \sim N_3(\underline{\mu}, \Sigma)$, $\underline{\mu} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & -3 & -8 \\ -3 & 2 & -6 \\ -8 & -6 & 3 \end{pmatrix}$

a) $E(\underline{y}^T \underline{A}\, \underline{y}) = tr(A\Sigma) + \mu^T A \mu = -1 + (-15) = -16$

b.) $var(y^T A y) = 2tr[(A\Sigma)^2] + 4\mu^T A \Sigma A \mu$ (Thm 5.2B in text)

$= 5782 + 15356 = 21{,}138.$

c) $y^T A y$ is not χ^2 because $A\Sigma$ is not idempotent

$A\Sigma = \begin{pmatrix} 1 & -13 & -27 \\ -10 & -5 & -16 \\ -38 & -17 & 3 \end{pmatrix} \neq A\Sigma A\Sigma = \begin{pmatrix} 1157 & 511 & 100 \\ 648 & 427 & 302 \\ 18 & 528 & 1307 \end{pmatrix}$

d.) No, because $\left(\frac{1}{\sigma^2} A\right)\Sigma = \frac{1}{\sigma^2} A \sigma^2 I = A$ is not idempotent

$AA = \begin{pmatrix} 74 & 39 & -14 \\ 39 & 49 & -6 \\ -14 & -6 & 109 \end{pmatrix} \neq A$

12. ~~Ex~~ (5.26) $\underline{y} \sim N_3(\underline{\mu}, \sigma^2 \underline{I})$, $\underline{\mu} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$, $\underline{A} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$

$\underline{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$

a) $\frac{1}{\sigma^2}\underline{y}^T \underline{A}\, \underline{y} \sim \chi^2(3, \underbrace{\frac{1}{2\sigma^2}\underline{\mu}^T \underline{A}\, \underline{\mu}})$

$$= \frac{1}{2\sigma^2}\,\frac{1}{3}\,(3,-2,1)\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$= \frac{1}{6\sigma^2}(7,-8,1)\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{6\sigma^2}\,38 = \frac{19}{3}\sigma^2$$

because A is idempotent

$$AA = \frac{1}{9}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = A$$

and A has rank 3 (full rank). Here, we've used Corollary 2 on p. ~~83~~ 84 of the notes.

b) Using corollary on p. ~~88~~ 89 of the notes, $\underline{y}^T \underline{A}\, \underline{y}$ and $\underline{B}\underline{y}$ are independent iff $\underline{B}\underline{A} = \underline{0}$.
Since

$$\underline{B}\,\underline{A} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & -3 \end{pmatrix} \neq \underline{0}$$

$\underline{y}^T \underline{A}\, \underline{y}$ and $\underline{B}\,\underline{y}$ are <u>not</u> independent.

c) $y_1 + y_2 + y_3 = (1,1,1)\,\underline{y}$ so must check whether $(1,1,1)\underline{A} = \underline{0}$

$(1,1,1)\,\frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = \frac{1}{3}(0,0,0) = \underline{0} \Rightarrow \underline{y}^T \underline{A}\, \underline{y}$ and $y_1 + y_2 + y_3$ are independent.