STAT 8260 — Theory of Linear Models
Homework 2 – Due Thursday, Feb. 7

Homework Guidelines:

- Homework is due by 4:30 on the due date specified above. You may turn it in at the beginning of class or place it in my mailbox in the Statistics Building. **No late homeworks will be accepted without permission granted prior to the due date.**
- Use only standard (8.5×11 inch) paper and use only one side of each sheet.
- Homework should show enough detail so that the reader can clearly understand the procedures of the solutions. This is **absolutely essential** for you to receive full credit for your answer since the answers to most of the problems in Rencher appear in the back of the book.
- Problems should appear in the order that they were assigned.
- Some problem numbers differ between the first and second editions of the book. When a problem number is given in parentheses, owners of the second edition should do the problem in parentheses, first edition user's should do the problem not in parentheses. E.g., if I assign problems 1,4, 7(8), 9(11), then first edition user's should do problems 1,4,7,9, and second edition user's should do problems 1,4,8,11.

Assignment:

1. Verify the result $C(\mathbf{X}^T\mathbf{X}) = C(\mathbf{X}^T)$ for

$$\mathbf{X} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 1 & 1 \end{pmatrix}.$$

2. For each of the following matrices or quadratic forms, determine whether it is positive definite, positive semidefinite, or neither:

   a. $Q(x_1, x_2, x_3) = 12x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 - 10x_1x_3 + 4x_2x_3$.

   b.

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 4 \end{pmatrix}$$

c.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 1 & -2 \end{pmatrix}$$

3. For $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ find a matrix $\mathbf{B}$ such that $\mathbf{A} = \mathbf{B}\mathbf{B}^T$.

4. Let

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & -1 & 1 \\ 3 & -3 & 4 \end{pmatrix}, \quad \mathbf{A}^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

Use these matrices to illustrate the following general result:

For $\mathbf{X}$ an $n \times k$ matrix of rank r and $\mathbf{X}^-$ its generalized inverse, then

$$\mathrm{tr}(\mathbf{X}^-\mathbf{X}) = \mathrm{tr}(\mathbf{X}\mathbf{X}^-) = \mathrm{rank}(\mathbf{X}^-\mathbf{X}) = \mathrm{rank}(\mathbf{X}\mathbf{X}^-) = \mathrm{rank}(\mathbf{X}).$$

5.–7. Do problem 2.31, 2.34(2.35), 2.38(2.39) of our text.

8. Compare the rank of the augmented matrix with the rank of the coefficient matrix for each of the following systems of equations. Find solutions where they exist.

a.

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ -x_1 - x_2 &= 1 \\ -x_1 + x_2 + 2x_3 &= 9 \end{aligned}$$

b.

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ -x_1 - x_2 &= 1 \\ x_2 + x_3 &= 9 \end{aligned}$$

c.

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 1 \\ -x_1 - x_2 + x_4 &= 2 \\ x_2 + x_3 - x_4 &= 1 \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= 1 \end{aligned}$$

9. For the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

and its generalized inverses

$$\mathbf{A}_1^- = \frac{1}{9}\begin{pmatrix} 1 & -5 & -4 \\ 2 & -1 & 1 \\ 1 & 4 & 5 \end{pmatrix}, \quad \text{and} \quad \mathbf{A}_2^- = \begin{pmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

show that $\mathbf{A}\mathbf{A}_1^-\mathbf{A} = \mathbf{A}\mathbf{A}_2^-\mathbf{A} = \mathbf{A}$.

10. For the matrix $\mathbf{A}$ of problem 9, show that the generalized inverse of $\mathbf{A}$ given by

$$\mathbf{A}_3^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 2 & 0 \end{pmatrix}$$

is a generalized inverse of $\mathbf{A}$ can be derived by the 5 step algorithm given by Corollary 1 following Theorem 2.8B of our text.

11.–12. Do problems 2.45(2.46) and 2.70(2.71) of our text. For problem 2.70(2.71) also do the following additional parts:

12.d. For $k = 3$, verify the property $\mathbf{A}^k\mathbf{v} = \lambda^k\mathbf{v}$ for each eigen-pair of $\mathbf{A}$.

12.e. Find $\text{tr}(\mathbf{A})$ and $|\mathbf{A}|$ and verify that $\text{tr}(\mathbf{A}) = \sum_{i=1}^{3} \lambda_i$ and $|\mathbf{A}| = \prod_{i=1}^{3} \lambda_i$ where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues