STAT 8260 Exam 1 – Tuesday, February 26
SHOW ALL WORK

Name: Key

1. **(12 pts)** Suppose that a matrix **M** is (i) symmetric, and (ii) idempotent. Prove that **M** is a projection matrix onto its column space $C(\mathbf{M})$.

Assume $M = M^T$, $MM = M$

show (a) if $v \in C(M)$ $Mv = v$

(b) if $v \perp C(M)$ $Mv = 0$

(a) if $v \in C(M)$ then $\underline{v} = \underline{M}\,\underline{b}$ for some $\underline{b} \neq 0$

$\Rightarrow$ $\underline{M}\,\underline{v} = \underline{M}\,\underline{M}\,\underline{b} = \underline{M}\,\underline{b} = \underline{v}$

(b) if $v \perp C(\underline{M})$ then $\underline{0} = \underline{M}^T\underline{v} = \underline{M}v$

2. **(10 pts)** Consider the system of equations

$$2x_1 + x_2 - x_3 = 1$$
$$4x_1 + 6x_2 + x_3 = 1$$
$$-4x_2 - 3x_3 = 2$$

Is this system of equations consistent? Justify your answer.

The system can be written as $\underline{A}\underline{x} = \underline{c}$ for $\underline{A} = \begin{pmatrix} 2 & 1 & -1 \\ 4 & 6 & 1 \\ 0 & -4 & -3 \end{pmatrix}$

$\underline{c} = (1,1,2)^T$, $\underline{x} = (x_1, x_2, x_3)^T$. Note that if $\underline{a}_i^T$ is the i^{th} row of $\underline{A}$, then $2\underline{a}_1 - \underline{a}_2 = \underline{a}_3$, so $\text{rank}(A) = 2$.

However $2c_1 - c_2 \neq c_3$ $(2 - 1 \neq 2)$, so the linear dependence in A does not hold in $[\underline{A}, \underline{c}]$ and $\text{rank}(A) = 2 \neq \text{rank}([A, c]) = 3$

$\Longrightarrow$ system is not consistent.

3. **(10 pts)** Suppose $\mathbf{y} = (y_1, y_2, y_3)^T \sim N_3(\mathbf{0}, \Sigma)$ where

$$\Sigma = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 4 & 1 \\ -1 & 1 & 9 \end{pmatrix}.$$

Find $\rho_{23 \cdot 1}$, the partial correlation between y_2 and y_3 adjusted for y_1.

$$\text{Var}\left(\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \Big| y_1 \right) = \begin{pmatrix} 4 & 1 \\ 1 & 9 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \end{pmatrix} (1)^{-1} (-1 \;\; -1)$$

$$= \begin{pmatrix} 4 & 1 \\ 1 & 9 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 8 \end{pmatrix}$$

$\Longrightarrow \rho_{23 \cdot 1} = 0$

4. ~~3.~~ Suppose an experiment is conducted in which four experimental units are randomized to two treatment groups of sizes 3 and 1, respectively. That is, four responses are observed, three under treatment 1, and one under treatment 2. Let y_{ij} denote the j^{th} observation in treatment group i, and let $\mathbf{y} = (y_{11}, y_{12}, y_{13}, y_{21})^T$. Define

$$V_0 = \mathcal{L}(\mathbf{j}_4) \qquad \text{and} \qquad V_1 = \mathcal{L}(\mathbf{i}_1, \mathbf{i}_2) \cap V_0^{\perp},$$

where $\mathbf{i}_1 = (1,1,1,0)^T$ and $\mathbf{i}_2 = (0,0,0,1)^T$.

a. (~~12~~ 14 **points**) Find $\mathbf{P}_{V_1}$ and $\mathbf{P}_{V_1^{\perp}}$, the projection matrices onto V_1 and its orthogonal completement $V_1^{\perp}$.

$$P_{V_1} = P_{\mathcal{L}(i_1, i_2)} - P_{V_0}$$

$$P_{\mathcal{L}(\underline{i}_1, \underline{i}_2)} = \begin{pmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P_{V_0} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \Rightarrow P_{V_1} = \begin{pmatrix} 1/12 & 1/12 & 1/12 & -\frac{1}{4} \\ 1/12 & 1/12 & 1/12 & -1/4 \\ 1/12 & 1/12 & 1/12 & -1/4 \\ -1/4 & -1/4 & -1/4 & 3/4 \end{pmatrix}$$

$$P_{V_1^{\perp}} = I - P_{V_1} = \begin{pmatrix} 11/12 & -1/12 & -1/12 & 1/4 \\ -1/12 & 11/12 & -1/12 & 1/4 \\ -1/12 & -1/12 & 11/12 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$$

b. **(14 points)** Suppose $\mathbf{y} \sim N_4(\boldsymbol{\mu}, \sigma^2 \mathbf{I}_4)$ where $\boldsymbol{\mu} = (\mu_1, \mu_1, \mu_1, \mu_2)^T$. Find the distribution of $\bar{y}_{1.}^2/(y_{21} - \mu_2)^2$ where $\bar{y}_{1.} = \frac{1}{3}(y_{11} + y_{12} + y_{13})$.

$$\bar{y}_{1.} \sim N(\mu_1, \sigma^2/3) \Rightarrow \frac{\sqrt{3}\,\bar{y}_{1.}}{\sigma} \sim N\left(\frac{\sqrt{3}\mu_1}{\sigma}, 1\right)$$

$$\Rightarrow \frac{3\bar{y}_{1.}^2}{\sigma^2} \sim \chi^2\left(1, \frac{3\mu_1^2}{2\sigma^2}\right)$$

$$y_{21} - \mu_2 \sim N(0, \sigma^2) \Rightarrow \frac{y_{21} - \mu_2}{\sigma} \sim N(0,1)$$

$$\Rightarrow \frac{(y_{21} - \mu_2)^2}{\sigma^2} \sim \chi^2(1)$$

Since $\bar{y}_{1.}^2$ is a function of y_{11}, y_{12}, y_{13} which are indep of y_{21}, $\bar{y}_{1.}^2$ is indep of $(y_{21} - \mu_2)^2$

so
$$\frac{\frac{3\bar{y}_{1.}^2}{\sigma^2}/1}{\frac{(y_{21}-\mu_2)^2}{\sigma^2}/1} \sim F\left(1, 1, \frac{3\mu_1^2}{2\sigma^2}\right)$$

$$= \frac{3\bar{y}_{1.}^2}{(y_{21}-\mu_2)^2}$$

$$\Rightarrow \frac{\bar{y}_{1.}^2}{(y_{21}-\mu_2)^2} \sim \frac{1}{3} F\left(1, 1, \frac{3\mu_1^2}{2\sigma^2}\right)$$

c. **(8 pts)** Find a matrix $\mathbf{B}$ such that $\mathcal{L}(\mathbf{i}_1, \mathbf{i}_2) \subset C(\mathbf{B}) \subset \mathcal{R}^4$. (Here, $W \subset U$ denotes that W is a proper subset of U; i.e., U contains W but is larger than W.)

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$ (many possible answers)

5. Let $\mathbf{y} \sim N_2(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu} = (\mu_1, \mu_2)^T$, and $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$ is a positive definite matrix. In addition, let $\mathbf{A} = \frac{1}{2}\mathbf{J}_{2,2}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$.

a. **(6 pts)** Prove or disprove: Any matrix of the form $c\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is positive definite.

I meant to specify $c > 0$

Notice $\left| c\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right| = c^2 - c^2 = 0$

$\Rightarrow c\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is singular $\Rightarrow$ not p.d.

b. **(12 pts)** Are $\mathbf{y}^T\mathbf{A}\mathbf{y} = 2\bar{y}^2$ and $\mathbf{B}\mathbf{y} = \begin{pmatrix} y_1 \\ y_1 - y_2 \end{pmatrix}$ independent? Why or why not?

Will be indep iff $B\Sigma A = 0$

$$B\Sigma A = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{11}-\sigma_{12} & \sigma_{12}-\sigma_{22} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \sigma_{11}+\sigma_{12} & \sigma_{11}+\sigma_{12} \\ \sigma_{11}-\sigma_{22} & \sigma_{11}-\sigma_{22} \end{pmatrix}$$ equals 0 iff

$$\sigma_{11}+\sigma_{12} = \sigma_{11}-\sigma_{22} = 0$$

$$\Rightarrow \Sigma = \begin{pmatrix} \sigma_{11} & -\sigma_{11} \\ -\sigma_{11} & \sigma_{11} \end{pmatrix} = \sigma_{11}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

which is ~~not~~ not p.d.

$\Rightarrow B\Sigma A \neq 0$ for any p.d. Σ

$\Rightarrow y^TAy, By$ are not independent

c. (~~12~~ 14 pts) Now assume that $\Sigma = \sigma^2\mathbf{I}$. What is the distribution of $\frac{2}{5\sigma^2}\mathbf{y}^T\mathbf{B}^T\mathbf{A}\mathbf{B}\mathbf{y}$?

$$\frac{2}{5\sigma^2} y^T B^T A B y \sim \chi^2(r, \lambda) \text{ where } r = \text{rank}\left(\frac{2}{5\sigma^2} B^T A B\right)$$

$$\text{and } \lambda = \frac{1}{2} \frac{2}{5\sigma^2} \mu^T B^T A B \mu$$

$$\text{iff } \underbrace{\frac{2}{5\sigma^2} B^T A B \sigma^2 I}$$ is idempotent

$$= \frac{2}{5} B^T A B = \frac{2}{5} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$$

$$\frac{1}{5} \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} = \frac{1}{25} \begin{pmatrix} 20 & -10 \\ -10 & 5 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$$

idempotent ✓

$$r = \text{rank}\left(\frac{2}{5\sigma^2} B^T A B\right) = \text{rank}\left(\frac{1}{5}\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\right) = 1$$ (2nd column is $-\frac{1}{2}$ times 1st column)

$$\lambda = \frac{1}{2\sigma^2} \mu^T \frac{1}{5} \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \mu$$

$$= \frac{1}{10\sigma^2} \mu^T \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \mu = \frac{1}{10\sigma^2}\left(4\mu_1^2 + \mu_2^2 - 4\mu_1\mu_2\right)$$