Robust Hierarchical Bayes Small Area Estimation for the Nested Error Linear Regression Model

Adrijo Chakraborty[1], Gauri Sankar Datta[2,3] and Abhyuday Mandal[2]

[1]*NORC at the University of Chicago, Bethesda, MD 20814, USA*

[2]*Department of Statistics, University of Georgia, Athens, GA 30602, USA*

[3]*Center for Statistical Research and Methodology, US Census Bureau*

E-mails: adrijo.chakraborty@gmail.com , gauri@stat.uga.edu and amandal@stat.uga.edu

Summary

Standard model-based small area estimates perform poorly in presence of outliers. Sinha and Rao (2009) developed robust frequentist predictors of small area means. In this article, we present a robust Bayesian method to handle outliers in unit-level data by extending the nested error regression model. We consider a finite mixture of normal distributions for the unit-level error to model outliers and produce noninformative Bayes predictors of small area means. Our modeling approach generalizes that of Datta and Ghosh (1991) under the normality assumption. Application of our method to a data set which is suspected to contain an outlier confirms this suspicion, correctly identifies the suspected outlier, and produces robust predictors and posterior standard deviations of the small area means. Evaluation of several procedures including the M-quantile method of Chambers and Tzavidis (2006) via simulations shows that our proposed method is as good as other procedures in terms of bias, variability and coverage probability of confidence and credible intervals when there are no outliers. In the presence of outliers, while our method and Sinha-Rao method perform similarly, they improve over the other methods. This superior performance of our procedure shows its dual

(Bayes and frequentist) dominance, which should make it attractive to all practitioners, Bayesians and frequentists, of small area estimation.

Key words:Normal mixture; outliers; prediction intervals and uncertainty; robust empirical best linear unbiased prediction; unit-level models.

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1 Introduction

The nested error regression (NER) model with the normality assumption for both the random effects or model error terms and the unit-level error terms has played a key role in analyzing unit-level data in small area estimation. Many popular small area estimation methods have been developed under this model. In the frequentist approach, Battese et al. (1988), Prasad and Rao (1990), and Datta and Lahiri (2000), for example, derived empirical best linear unbiased predictors (EBLUPs) of small area means. These authors used various estimation methods for the variance components and derived approximately accurate estimators of mean squared error (MSEs) of the EBLUPs. On the other hand, Datta and Ghosh (1991) followed the hierarchical Bayesian (HB) approach to derive posterior means as HB predictors and variances of the small area means. While the underlying normality assumptions for all the random quantities are appropriate for regular data, they fail to adequately accommodate outliers. Consequently, these frequentist/Bayesian methods are highly influenced by major outliers in the data, or break down if the outliers grossly violate distributional assumptions.

Sinha and Rao (2009) investigated the robustness, or lack thereof, of the EBLUPs from the usual normal NER model in the presence of "representative outliers". According to Chambers (1986), a representative outlier is a "sample element with a value that has

been correctly recorded and cannot be regarded as unique. In particular, there is no reason to assume that there are no more similar outliers in the nonsampled part of the population." Sinha and Rao (2009) showed via simulations for the NER model that while the EBLUPs are efficient under normality, they are very sensitive to outliers that deviate from the assumed model.

To address the non-robustness issue of EBLUPs, Sinha and Rao (2009) used the ψ-function, Huber's Proposal 2 influence function in M-estimation, to downweight the contribution of outliers in the BLUPs and the estimators of the model parameters, both regression coefficients and variance components. Using M-estimation for robust maximum likelihood, estimators of model parameters, and robust predictors of random effects, Sinha and Rao (2009) for mixed linear models proposed a robust EBLUP (REBLUP) of mixed effects, which they used to estimate small area means for the NER model. By using a parametric bootstrap procedure they have also developed estimators of the MSEs of the REBLUPs. We refer to Sinha and Rao (2009) for details of this method. Their simulations show that when the normality assumptions hold, the proposed REBLUPs perform similar to the EBLUPs in terms of empirical bias and empirical MSE. But, in presence of outliers in the unit-level errors, while both EBLUPs and REBLUPs remain approximately unbiased, the empirical MSEs of the EBLUPs are significantly larger than those of the REBLUPs.

Datta and Ghosh (1991) proposed a noninformative HB model to predict finite population small area means. In this article we follow the approach to finite population sampling which was also followed by Datta and Ghosh (1991). Our suggested model includes the treatment of the NER model by Datta and Ghosh (1991) as a special case. Our model facilitates accommodating outliers in the population and in the sample values. We replace the normality of the unit-level error terms by a two-component mixture of normal distributions, each component centered at zero. As in Datta and Ghosh (1991), we assume normality of the small area effects.

Simulation results of Sinha and Rao (2009) indicated that there was not enough im-

provement in performance of the REBLUP procedures over the EBLUPs when they considered outliers in both the unit-level error and the model error terms. To keep both analytical and computational challenges for our noninformative HB analysis manageable, we use a realistic framework and we restrict ourselves to the normality assumption for the random effects. Moreover, the assumption of zero means for the unit-level error terms is similar to the assumption made by Sinha and Rao (2009). While allowing the component of the unit-level error terms with the bigger variance to also have non-zero means to accommodate outliers might appear attractive, we note later that it is not possible to conduct a noninformative Bayesian analysis with an improper prior on the new parameter.

We focus only on unit-level model robust small area estimation in this article. There is a substantial literature on small area estimation based on area-level data using the Fay-Herriot model (see Fay and Herriot, 1979; Prasad and Rao, 1990). The paper by Sinha and Rao (2009) also discussed robust small area estimation for an area-level model. In another paper, Lahiri and Rao (1995) discussed EBLUP and estimation of MSE under a non-normality assumption for the random effects. An early robust Bayesian approach for area-level models is due to Datta and Lahiri (1995), where they used a scale mixture of normal distributions for the random effects. It is worth mentioning that the t-distributions are special cases of the scale mixture of normal distributions. While Datta and Lahiri (1995) assumed long-tailed distributions for the random effects, Bell and Huang (2006) used the HB method based on the t distribution, either only for the sampling errors or only for the model errors.

The scale mixture of normal distributions requires specification of the mixing distribution, or in the specific case for t distributions, it requires the degrees of freedom. In an attempt to avoid this specification, in a recent article Chakraborty et al. (2016) proposed a simple alternative via a two-component mixture of normal distributions in terms of the variance components for the model errors.

2 Unit-Level HB Models for Small Area Estimation

The model-based approach to finite population sampling is very useful for modeling unit-level data in small area estimation. The NER model of Battese et al. (1988) is a popular model for unit-level data. Suppose a finite population is partitioned into m small areas, with the ith area having N_i units. The NER model relates Y_{ij}, the value of a response variable Y for the jth unit in the ith small area, with $x_{ij} = (x_{ij1}, \cdots, x_{ijp})^T$, the value of a p-component covariate vector associated with that unit, through a mixed linear model given by

$$Y_{ij} = x_{ij}^T \beta + v_i + e_{ij},\ j = 1, \cdots, N_i,\ i = 1, \cdots, m, \tag{2.1}$$

where all the random variables v_i's and e_{ij}'s are assumed independent. Distributions of these variables are specified by assuming that random effects $v_i \overset{iid}{\sim} N(0, \sigma_v^2)$ and unit-level errors $e_{ij} \overset{iid}{\sim} N(0, \sigma_e^2)$. Here $\beta = (\beta_1, \cdots, \beta_p)^T$ is the regression coefficient vector. We want to predict the ith small area finite population mean $\bar{Y}_i = N_i^{-1} \sum_{j=1}^{N_i} Y_{ij}$, $i = 1, \cdots, m$. We assume that the population level model (2.1) holds for any sample from the population.

Battese et al. (1988) and Prasad and Rao (1990), among others, considered noninformative sampling, where a simple random sample of size n_i is selected from the ith small area. For notational simplicity we denote the sample by $Y_{ij}, j = 1, \cdots, n_i, i = 1, \cdots, m$. To develop predictors of the small area means $\bar{Y}_i, i = 1, \cdots, m$, these authors first derived, for known model parameters, the conditional distribution of the *unsampled* values, $Y_{ij}, j = n_i + 1, \cdots, N_i, i = 1, \cdots, m$, given the sampled values $Y_{ij}, j = 1, \cdots, n_i, i = 1, \cdots, m$. Under squared error loss, the best predictor of $\bar{Y}_i$ is its mean with respect to this conditional distribution, also known as the predictive distribution. In the frequentist approach, Battese et al. (1988) and Prasad and Rao (1990) obtained the EBLUP of $\bar{Y}_i$ by replacing in the conditional mean the unknown model parameters $(\beta^T, \sigma_e^2, \sigma_v^2)^T$ by their estimators using $Y_{ij}, j = 1, \cdots, n_i, i = 1, \cdots, m$. In the Bayesian approach, on the other hand, Datta and Ghosh (1991) developed HB predictors of $\bar{Y}_i$ by integrating out these parameters in the conditional mean of $\bar{Y}_i$ with respect to their posterior density, which is derived based on a prior distribution on the parameters and the distribution of

the sample $Y_{ij}, j = 1, \cdots, n_i, i = 1, \cdots, m$, derived under the model (2.1).

While the frequentist approach for the NER model under the distributional assumptions in (2.1) continues with accurate approximation and estimation of the MSEs of the EBLUPs, the Bayesian approach typically proceeds under some noninformative priors, and computes numerically, usually by the MCMC method, the exact posterior means and posterior variances of the area means $\bar{Y}_i$'s. Among various noninformative priors for $\beta, \sigma_e^2, \sigma_v^2$, a popular choice is

$$\pi_P(\beta, \sigma_e^2, \sigma_v^2) = \frac{1}{\sigma_e^2}, \tag{2.2}$$

(see, for example, Datta and Ghosh, 1991).

The standard NER model in (2.1) is unable to explain outlier behavior of unit-level error terms. To avoid the breakdown of EBLUPs and their MSEs in the presence of outliers, Sinha and Rao (2009) modified all estimating equations for the model parameters and random effects terms by robustifying various "standardized residuals" that appear in the estimating equations by using Huber's ψ-function, which truncates large absolute values to a certain threshold. They did not replace the working NER model in (2.1) to accommodate outliers, but they accounted for their potential impacts on the EBLUPs and estimated MSEs by downweighting large standardized residuals that appear in various estimating equations through Huber's ψ-function. Their approach, in the terminology of Chambers et al. (2014), may be termed *robust projective*, where they estimated the working model in a robust fashion and used that to project sample non-outlier behavior to the unsampled part of the model.

To investigate the effectiveness of their proposal, Sinha and Rao (2009) conducted simulations based on various long-tailed distributions for the random effects and/or the unit-level error terms. In one of their simulation scenarios which is reasonably simple but useful, they used a two-component mixture of normal distributions for the unit-level error terms, with both components centered at zero but with unequal variances, and the component with the larger variance appearing with a small probability. This modifies the regular setup of the NER model with the possibility of outliers arising as a small fraction

of contamination caused by the error corresponding to the larger variance component. Simulation results in Table 2 of Sinha and Rao (2009) report that outliers in the random effect have little impact on the EBLUP. Hence we could focus on the unit-level error only. In this article, we incorporate this mixture distribution to modify the model in (2.1) to develop new Bayesian methods that would be robust to outliers. Our proposed population level HB model is given by

Normal Mixture (NM) HB Model:

(I) Conditional on $\beta = (\beta_1, \cdots, \beta_p)^T, v_1, \cdots, v_m, z_{ij}, j = 1, \cdots, N_i, i = 1, \cdots, m, p_e, \sigma_1^2, \sigma_2^2$ and σ_v^2,

$$Y_{ij} \overset{ind}{\sim} z_{ij} N(x_{ij}^T \beta + v_i, \sigma_1^2) + (1 - z_{ij}) N(x_{ij}^T \beta + v_i, \sigma_2^2),\ j = 1, \cdots, N_i, i = 1, \cdots, m.$$

(II) The indicator variables z_{ij}'s are iid with $P(z_{ij} = 1 | p_e) = p_e,\ j = 1, \cdots, N_i, i = 1, \cdots, m$, and are independent of $\beta = (\beta_1, \cdots, \beta_p)^T, v_1, \cdots, v_m, \sigma_1^2, \sigma_2^2$ and σ_v^2.

(III) Conditional on $\beta, z = (z_{11}, \cdots, z_{1N_1}, \cdots, z_{m1}, \cdots, z_{mN_m})^T, p_e, \sigma_1^2, \sigma_2^2$ and σ_v^2, random small area effects $v_i \overset{iid}{\sim} N(0, \sigma_v^2)$ for $i = 1, \cdots, m$.

For simplicity, we assume the contamination probability p_e to remain the same for all units in all small areas. Gershunskaya (2010) proposed this mixture model for empirical Bayes point estimation of small area means. We assume independent simple random samples of size $n_1, \cdots, n_m$ from the m small areas. The Simple Random Sampling results in a noninformative sample and the joint distribution of responses of the sampled units can be obtained from the NM HB model above by replacing N_i by n_i. This marginal distribution in combination with the prior distribution provided below will yield the posterior distribution of the v_i's, and of all the parameters in the model. For the informative sampling developments in small area estimation we refer to Pfeffermann and Sverchkov (2007) and Verret et al. (2015).

Two components of the normal mixture distribution differ only by their variances. We will assume the variance component σ_2^2 is larger than σ_1^2 and is intended to explain any

outliers in a data set. However, if a data set does not include any outliers, the two component variances σ_1^2, σ_2^2 may only minimally differ. In such situation, the likelihood based on the sample will include limited information to distinguish between these variance parameters, and consequently, the likelihood will also have little information about the mixing proportion p_e. We notice this behavior in our application to a subset of the corn data in Section 5.

In this article, we carry out an objective Bayesian analysis by assigning a noninformative prior to the model parameters. In particular, we propose a noninformative prior

$$\pi(\beta, \sigma_1^2, \sigma_2^2, \sigma_v^2, p_e) = \frac{I(0 < \sigma_1^2 < \sigma_2^2 < \infty)}{(\sigma_2^2)^2}, \tag{2.3}$$

where we have assigned an improper prior on $\beta, \sigma_v^2, \sigma_1^2, \sigma_2^2$ and a proper uniform prior on the mixing proportion p_e. However, subjective priors could also be assigned when such subjective information is available. Notably, it is possible to use some other proper prior on p_e that may elicit the extent of contamination to the basic model to reflect prevalence of outliers. While many such subjective priors can be reasonably modeled by a beta distribution, we use a *uniform distribution* from this class to reflect noninformativeness or little information about this parameter. We also use the traditional uniform priors on β and σ_v^2. In the Supplementary materials, we explore the propriety of the posterior distribution corresponding to the improper priors in (2.3).

The improper prior distribution on the two variances for the mixture distribution has been carefully chosen so that the prior will yield conditionally proper distributions for each parameter given the other. The proper conditional densities given σ_2^2 (or σ_1^2) respectively are

$$\pi(\sigma_1^2|\sigma_2^2) = \frac{1}{\sigma_2^2} I(0 < \sigma_1^2 < \sigma_2^2), \quad \pi(\sigma_2^2|\sigma_1^2) = \frac{\sigma_1^2}{(\sigma_2^2)^2} I(\sigma_1^2 < \sigma_2^2 < \infty).$$

This conditional propriety is *necessary* for parameters appearing in the mixture distribution in order to ensure under suitable conditions the propriety of the posterior density resulting from the HB model. Alternatively, if we used, $\pi(\sigma_1^2, \sigma_2^2) \propto (\sigma_1^2)^{-1}(\sigma_2^2)^{-1}$, the posterior distribution would be improper for situations when there are no observations

from the outlying distribution. Prior (2.3) can accommodate these situations. The specific prior distribution that we propose above is such that the resulting marginal densities for σ_1^2 and σ_2^2 respectively, are $\pi_{\sigma_1^2}(\sigma_1^2) = (\sigma_1^2)^{-1}$ and $\pi_{\sigma_2^2}(\sigma_2^2) = (\sigma_2^2)^{-1}$. These two densities are of the same form as that of σ_e^2 in the regular model in (2.2) introduced earlier. Indeed by setting $p_e = 0$ or 1 in our analysis, we can reproduce the HB analysis of the regular model given by (2.1) and (2.2).

We use the NM HB Model under noninformative sampling and the noninformative priors given by (2.3) to derive the posterior predictive distribution of $\bar{Y}_i, i = 1, \cdots, m$. The NM HB model and noninformative sampling that we propose here facilitate building model for *representative outliers* (Chambers, 1986). According to Chambers, a representative outlier is a value of a sampled unit which is not regarded as unique in the population, and one can expect existence of similar values in the non-sampled part of the population which will influence the value of the finite population means $\bar{Y}_i$'s or the other parameters involved in the superpopulation model.

Following the practice of Battese et al. (1988) and Prasad and Rao (1990), we approximated the predictand $\bar{Y}_i$ by $\theta_i = \bar{X}_i^T\beta + v_i$ to draw inference on the finite population small area means. Here $\bar{X}_i = N_i^{-1}\sum_{j=1}^{N_i} x_{ij}$ is assumed known. This approximation works well for small sampling fractions n_i/N_i and large N_i's. It has been noted by these authors, and by Sinha and Rao (2009), that even for the case of outliers in the sample the difference between the inference results for $\bar{Y}_i$ and θ_i is negligible. Our own simulations for our model also confirm that observation. Once MCMC samples from the posterior distribution of β, v_i's and $\sigma_v^2, \sigma_1^2, \sigma_2^2, p_e$ have been generated, using the NM HB Model the MCMC samples of $Y_{ij}, j = n_i + 1, \cdots, N_i, i = 1, \cdots, m$ from their posterior predictive distributions can be easily generated. Finally, using the relation $\bar{Y}_i = N_i^{-1}[\sum_{j=1}^{n_i} y_{ij} + \sum_{j=n_i+1}^{N_i} Y_{ij}]$, (posterior predictive) MCMC samples for $\bar{Y}_i$'s can be easily generated for inference on these quantities. In our own data anaysis, where the sampling fractions are negligible, we do inference for the approximated predictands θ_i's.

Chambers and Tzavidis (2006) took a new frequentist approach to small area estimation

that is different from the mixed model prediction used in EBLUP. Instead of using a mixed model for the response, they suggeted a method based on quantile regression. We briefly review their M-quantile small area estimation method in Section 3. They also proposed an estimator of MSE of their point estimators.

Our Bayesian proposal has two advantages over the REBLUP of Sinha and Rao (2009). First, instead of a working model for the non-outliers, we use an explicit mixture model to specify the joint distribution of responses of all the units in the population, and not only the non-outliers part of the population. It enables us to use all the sampled observations to predict the entire non-sampled part, consisting of outliers and non-outliers, of the population. Our method is robust predictive and the noninformative HB predictors are less susceptible to bias. Second, the main thrust of the EBLUP approach in small area estimation is to develop accurate approximations and estimation of MSEs of EBLUPs (cf. Prasad and Rao, 1990). Datta and Lahiri (2000) and Datta et al. (2005) termed this approximation as second-order accurate approximation, which neglects terms lower order than m^{-1} in the approximation. Second-order accurate approximation results for REBLUPs have not been obtained by Sinha and Rao (2009). Also, their bootstrap proposal to estimation of the MSE under the working model has not been shown to be second-order accurate. Our HB proposal does not rely on any asymptotic approximations. Analysis of the corn data set and simulation study show less uncertainty (and better stability of this measure) of our method compared to the M-quantile method.

3 M-quantile Small Area Estimation

Small area estimation is dominated by linear mixed effects models where the conditional mean of Y_{ij}, the response of the jth unit in the ith small area, is expressed as $E(Y_{ij}|x_{ij}, v_i) = x_{ij}^T\beta + z_{ij}^T v_i$, where x_{ij} and z_{ij} are suitable known covariates, v_i is a random effects vector and β is a common regression coefficient vector. This assumption is the building block for EBLUPs of small area means, based on suitable additional assumptions for this conditional distribution and the distribution of the random effects.

Also with suitable prior distribution on the model parameters, HB methodology for prediction of small area means is developed.

As an alternative to linear regression which models $E(Y|x)$, the mean of the conditional distribution of Y given covariates x, quantile regression has been developed by modeling suitable quantiles of the conditional distribution of Y given x. In particular in quantile linear regression, for $0 < q < 1$, the qth quantile $Q_q(Y|x)$ of this distribution is modeled as $Q_q(Y|x) = x^T\beta_q$, where β_q is a suitable parameter modeling the linear quantile function. For a given quantile regression function, the quantile coefficient $q_i \in (0,1)$ of an observation y_i satisfies $Q_{q_i}(Y|x_i) = y_i$. In particular, for a linear quantile function, for given y_i, x_i, the q_i satisfies $x_i^T\beta_{q_i} = y_i$.

While in the linear regression setup the regression coefficient β is estimated from a set of data $\{y_i, x_i : i = 1, \cdots, n\}$ by minimizing the sum of squared errors $\sum_{i=1}^n (y_i - x_i^T\beta)^2$ with respect to β, the quantile regression coefficient β_q for a fixed $q \in (0,1)$ is obtained by minimizing the loss function $\sum_{i=1}^n |y_i - x_i^T b|\{(1-q)I(y_i - x_i^T b \leq 0) + qI(y_i - x_i^T b > 0)\}$ with respect to b. Here $I(\cdot)$ is a usual indicator function.

Following the idea of M-estimation in robust linear regression, Breckling and Chambers (1988) generalized quantile regression by minimizing an objective function $\sum_{i=1}^n d(|y_i - x_i^T b|)\{(1-q)I(y_i - x_i^T b \leq 0) + qI(y_i - x_i^T b > 0)\}$ with respect to b for some given loss function $d(\cdot)$. [Linear regression is a special case for $q = .5$ and $d(u) = u^2$.] Estimator of β_q is obtained by solving the equation

$$\sum_{i=1}^n \psi_q(r_{iq}) x_i = 0,$$

where $r_{iq} = y_i - x_i^T\beta_q$, $\psi_q(r_{iq}) = \psi(s^{-1}r_{iq})\{(1-q)I(r_{iq} \leq 0) + qI(r_{iq} > 0)\}$, the function $\psi(\cdot)$, known as the influence function in M-estimation, is determined by $d(\cdot)$ (actually, $\psi(u)$ is related to the derivative of $d(u)$, assuming it is differentiable). The quantity s is a suitable scale factor determined from the data (cf. Chambers and Tzavidis, 2006). In M-quantile regression, these authors suggested using $\psi(\cdot)$ as the Huber Proposal 2

influence function $\psi(u) = uI(|u| \leq c) + c\ \text{sign}(u)I(|u| > c)$, where c is a given positive number bounded away from 0.

To apply the M-quantile method in small area estimation for a set of data $\{y_{ij}, x_{ij}, j = 1, \cdots, n_i, i = 1, \cdots, m\}$, Fabrizi et al. (2012) followed Chambers and Tzavidis (2006) and suggested determining a set of $\hat{\beta}_q$ in a fine grid for $q \in (0, 1)$ by solving

$$\sum_{i=1}^{m} \sum_{j=1}^{n_i} \psi_q(r_{ijq}) x_{ij} = 0,$$

where $r_{ijq} = y_{ij} - x_{ij}^T \hat{\beta}_q$. Fabrizi et al. (2012) defined M-quantile estimator of $\bar{Y}_i$ by

$$\hat{\bar{Y}}_{i,MQ} = \frac{1}{N_i}[\sum_{j=1}^{n_i} y_{ij} + \sum_{j=n_i+1}^{N_i} x_{ij}^T \hat{\beta}_{\bar{q}_i} + (N_i - n_i)(\bar{y}_i - \bar{x}_i^T \hat{\beta}_{\bar{q}_i})], \tag{3.1}$$

where $(\bar{y}_i, \bar{x}_i)$ is the sample mean of $\{(y_{ij}, x_{ij}), j = 1, \cdots, n_i\}$. Here $\bar{q}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} q_{ij}$ is the average estimated quantile coefficient of the ith small area, where q_{ij} is obtained by solving $x_{ij}^T \hat{\beta}_q = y_{ij}$, based on the set $\{\hat{\beta}_q\}$ described above (if necessary, interpolation for q is made to solve $x_{ij}^T \hat{\beta}_q = y_{ij}$ accurately). Here we suppress the dependence of $\hat{\beta}_q$ and q_{ij} on the influence function $\psi(\cdot)$. For details on M-quantile small area estimators and associated estimators of MSE based on a pseudo-linearization method, we refer to Tzavidis and Chambers (2005) and Chambers et al. (2014).

4 Robust Empirical Best Linear Unbiased Prediction

Empirical best linear unbiased predictors (EBLUPs) of small area means, developed under normality assumptions for the random effects and the unit-level errors, play a very useful role in production of reliable model-based estimation methods. While the EBLUPs are efficient under the normality assumptions, they may be highly influenced by outliers in the data. Sinha and Rao (2009) investigated the robustness of the classical EBLUPs to the departure from normality assumptions and proposed a new class of predictors which are resistant to outliers. Their proposed robust modification of EBLUPs of small area means, which they termed robust EBLUP (REBLUP), downweight any influential observations in the data in estimating the model parameters and the random effects.

Sinha and Rao (2009) considered a general linear mixed effects model with a block-diagonal variance-covariance matrix. Their model, which is sufficiently general to include the popular Fay-Herriot model and the nested error regression model as special cases, is given by

$$y_i = X_i\beta + Z_i v_i + e_i, i = 1, \cdots, m, \tag{4.1}$$

for specified design matrices X_i, Z_i, random effects vector v_i and unit-level error vector e_i associated with the data y_i from the ith small area. They assumed normality and independence of the random vectors $v_1, \cdots, v_m, e_1, \cdots, e_m$, where $v_i \sim N(0, G_i(\delta))$ and $e_i \sim N(0, R_i(\delta))$. Here δ includes the variance parameters associated with the model (4.1).

To develop a robust predictor of a mixed effect $\mu_i = h_i^T\beta + k_i^T v_i$, Sinha and Rao (2009) started with the well-known mixed model equations given by

$$\sum_{i=1}^{m} X_i^T R_i^{-1}(y_i - X_i\beta - Z_i v_i) = 0,\ Z_i^T R_i^{-1}(y_i - X_i\beta - Z_i v_i) - G_i^{-1} v_i = 0,\ i = 1, \cdots, m, \tag{4.2}$$

which are derived as estimating equations by differentiating the joint density of $y_1, \cdots, y_m$, and $v_1, \cdots, v_m$ with respect to β, and $v_1, \cdots, v_m$ to obtain"maximum likelihood" estimators of $\beta, v_1, \cdots, v_m$ for known δ. The unique solution $\tilde{\beta}(\delta), \tilde{v}_1(\delta), \cdots, \tilde{v}_m(\delta)$ to these equations leads to the BLUP $h_i^T\tilde{\beta} + k_i^T\tilde{v}_i$ of μ_i. To estimate the variance parameters δ, Sinha and Rao (2009) maximized the profile likelihood of δ, which is the value of the likelihood of β and δ based on the joint distribution of the data $y_1, \cdots, y_m$ at $\beta = \tilde{\beta}(\delta)$.

To mitigate the impact of outliers on the estimators of the variance parameters, the regression coefficients and the random effects, Sinha and Rao (2009) extended the work of Fellner (1986) to robustify all the "estimating equations" by using Huber's ψ-function in M-estimation. Based on the robustified estimating equations, Sinha and Rao (2009) obtained the robust estimators of β, δ and $v_i, i = 1, \cdots, m$, denoted respectively by $\hat{\beta}_M, \hat{\delta}_M$ and $\hat{v}_{iM}, i = 1, \cdots, m$. These estimators lead to the REBLUP of μ_i given by $h_i^T\hat{\beta}_M + k_i^T\hat{v}_{iM}$. For details of the REBLUP and the associated parametric bootstrap estimators of the MSE of the REBLUPs of μ_i, we refer the readers to the paper by Sinha and Rao (2009).

5 Data Analysis

We illustrate our method by analyzing the crop areas data reported by Battese et al. (1988) who considered EBLUP prediction of county crop areas for 12 counties in Iowa. Based on U.S. farm survey data in conjunction with LANDSAT satellite data they developed predictors of county means of hectares of corn and soybeans. Battese et al. (1988) were the first to put forward the nested error regression model for the prediction of the county crop areas. Datta and Ghosh (1991) later used the HB prediction approach on this data to illustrate Bayesian treatment of the nested error regression model. In the USDA farm survey data on 37 sampled segments from these 12 counties, Battese et al. (1988) determined in their reported data that the second observation for corn in Hardin county was an outlier so that this outlier would not unduly affect the model-based estimates of the small area means, Battese et al. (1988) initially recommended, and Datta and Ghosh (1991) subsequently followed, to remove this suspected outlier observation from their analyses. Discarding this observation results in a better fit for the nested error regression model. However, removing any data which may not be a non-representative outlier from analysis will result in loss of valuable information about a part of the non-sampled units of the population which may contain outliers.

Table 1: Various point estimates and standard errors of county hectares of corn

SA	Full Data									Reduced Data								
	n_i	DG HB		NM HB		SR		MQ		n_i	DG HB		NM HB		SR		MQ	
		Mean	SD	Mean	SD	Mean	SD	Mean	SD		Mean	SD	Mean	SD	Mean	SD	Mean	SD
1	1	123.8	11.7	123.4	9.8	123.7	9.9	130.0	5.7	1	122.0	11.6	121.7	9.7	122.2	9.9	128.0	3.7
2	1	124.9	11.4	126.6	10.3	125.3	9.7	134.2	8.4	1	126.4	10.9	127.2	9.7	126.5	9.5	133.4	6.0
3	1	110.0	12.3	108.0	11.3	110.3	9.4	86.0	18.3	1	107.6	12.4	105.6	10.1	106.7	9.5	94.6	14.4
4	2	114.2	10.7	112.3	10.2	114.1	8.8	114.4	3.4	2	108.9	10.5	108.2	8.7	111.0	8.3	113.3	3.7
5	3	140.3	10.8	142.1	8.1	140.8	7.8	144.2	11.3	3	143.6	9.7	144.1	7.0	143.3	7.1	144.2	9.3
6	3	110.0	9.6	111.4	7.6	110.8	7.6	108.6	3.9	3	112.3	9.7	112.5	6.5	112.3	7.1	114.5	5.4
7	3	116.0	9.7	114.3	7.6	115.2	7.3	116.3	4.2	3	113.4	9.1	112.5	6.8	112.9	7.1	115.4	3.8
8	3	123.2	9.5	122.7	7.9	122.7	7.5	122.5	3.9	3	121.9	8.8	121.9	6.6	121.9	7.1	122.7	4.0
9	4	112.6	9.9	113.9	6.9	113.5	6.5	115.3	5.8	4	115.5	9.2	115.7	5.7	115.3	6.4	115.7	4.6
10	5	124.4	8.9	123.5	6.1	124.1	6.3	121.6	4.7	5	124.8	8.4	124.4	5.4	124.5	5.3	123.1	4.0
11	5	111.3	8.9	108.2	6.8	109.5	6.2	106.9	10.6	5	107.7	8.5	106.3	5.7	106.8	5.4	105.5	7.0
12	6	130.7	8.3	135.3	7.5	136.9	6.0	135.8	4.3	5	142.6	9.0	143.5	5.9	143.1	5.8	140.6	4.9

We reanalyze the full data set for corn using our proposed HB method as well as the other methods we reviewed above. In Table 1 we report various point estimates and standard error estimates. We compare our proposed robust HB prediction method with the standard HB method of Datta and Ghosh (1991), and two robust frequentist methods, the REBLUP method of Sinha and Rao (2009) and the MQ method of Chambers and Tzavidis (2006). We list in the table various estimates of county hectares of corn, along with their estimated standard errors or posterior standard deviations. Our analysis of the full data set including the potential outlier from the last small area shows that for the first 11 small areas there is a close agreement among the three sets of point estimates by Datta and Ghosh (1991), Sinha and Rao (2009) and the proposed normal mixture HB method. The Datta and Ghosh method, which was not developed to handle outliers, yields a point estimate for the 12th small area that is much different from the point estimates from Sinha-Rao or the proposed NM HB method. The latter two robust estimates are very similar in terms of point estimates for all the small areas. But when we compare these two sets of robust estimates with those from another robust method, namely, the MQ estimates, we find that the MQ estimates for the first three small areas are widely different from those for the other two methods. These numbers possibly indicate a potential bias of the MQ estimates.

To compare performance of all these methods in the absence of any potential outliers, we reanalyzed the corn data by removing the suspected outlier (our robust HB analysis confirmed the outlier status of this observation, cf. Figure 1 below). When we compare the MQ estimates with the four other sets of estimates, the DG HB, the SR, the NM, which are reported in Table 1, and the EB estimates from Table 3 of Fabrizi et al. (2012), we notice a great divide between the MQ estimates and the other estimates. Out of the twelve small areas, the estimates for areas 1, 2, 3, 5, and 6 from the MQ method differ substantially from the estimates from the other four methods. On the other hand, the close agreement among the last four sets of estimates also shows in general the usefulness of the robust predictors, the proposed HB predictors and the Sinha-Rao robust EBLUP predictors.

To examine the influence of the outlier on the estimates we compare changes in the estimates from both the full and reduced data. We find that the largest change occurs, not surprisingly, for the DG HB method for the small area suspected of the outlier. Such a large change occurred since the DG method cannot suitably downweight an outlier, consequently, it treated the outlier value of 88.59 in the same manner as it treated any other non-outlier observation. As a result, the predictor substantially underestimated the true mean $\bar{Y}_i$ for Hardin county. The next largest difference occurred for the MQ method for small area 3 which is not known to include any outlier. Such a large change is contrary to behavior of a robust method.

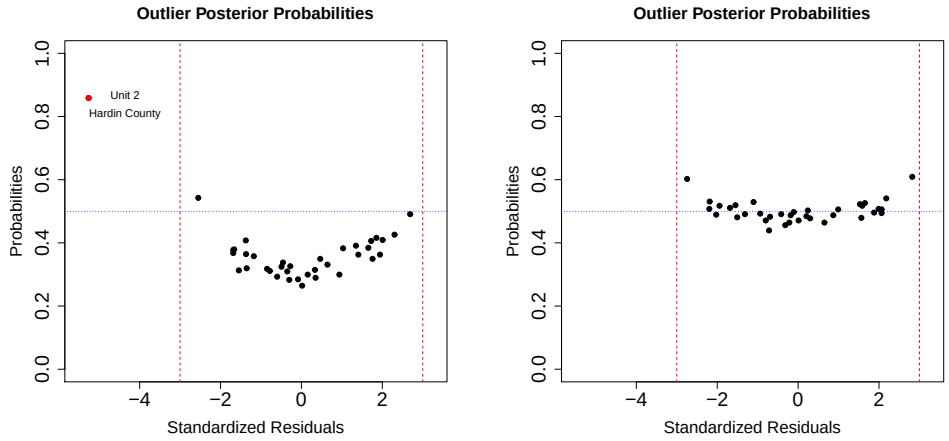

Figure 1: Posterior probabilities of observations being outliers in *full* and *reduced* data

The changes in point estimates for the robust HB and the REBLUP methods are moderate for the areas not known to include any outliers, and the changes seem proportionate for the small area suspected of an outlier. The corresponding changes in the estimates from the MQ method for some of the areas not including any outlier seem disproportionately large, and the change in the estimate for the area suspected of an outlier is not as large. This behavior to some extent indicates a lack of robustness of the MQ method to outliers.

An inspection of the posterior standard deviations of the two Bayesian methods reveals some interesting points. First, the posterior SDs of the small area means for the proposed

mixture model appear to be substantially smaller than the posterior SDs associated with the Datta-Ghosh HB estimators. Smaller posterior SDs suggest the posterior distribution of the small area means under the mixture model are more concentrated than those under the Datta-Ghosh model. This has been confirmed by simulation study, reported in the next section.

Next, when we compare the posterior SDs of small area means for our proposed method based on the full data and the reduced data, all posterior SDs increase for the full data (which likely contain an outlier). In the presence of outliers, the unit-level variance is expected to be large. Even though the posterior SDs of the small area means do not depend entirely only on the unit-level error variance, they are expected to increase with this variance. This monotonic increase appears reasonable due to the suspected outlier. While this intuitive property holds for our proposed method, it does not hold for the standard Datta-Ghosh method.

For further demonstration of the effectiveness of our proposed robust HB method, we computed model parameter estimates for both the reduced and the full data sets. These estimates are displayed in Table 2. The HB estimate of the larger variance component (976, based on mean) of the mixture is much larger than the estimate of the smaller component (182) for the full data, indicating a necessity of the mixture model. On the other hand, for the reduced data the estimates of variances for the two mixing components, 231 and 121, respectively are very similar and can be argued identical within errors in estimation, indicating limited need of the mixture distribution. A comparison of the estimates of p_e for the two cases also reveals the appropriateness of the mixture model for the full data. It also shows the redundancy of including p_e in the modeling of the reduced data as explained below.

The posterior density in a reasonable noninformative Bayesian analysis is usually dominated by the likelihood of the parameters generated by the data. In case the data do not provide much information about some parameters to the likelihood, posterior densities of such parameters will be dominated by their prior information. Consequently,

Table 2: Parameter estimates for various models with and without the suspected outlier

Estimates	Datta-Ghosh HB		Datta-Ghosh HB		Proposed Mixture HB		Proposed Mixture HB		Sinha-Rao	
Estimates	MEAN		MEDIAN		MEAN		MEDIAN		Sinha-Rao	
	Full Data	Reduced Data	Full Data	Reduced Data	Full Data	Reduced Data	Full Data	Reduced Data	Full Data	Reduced Data
$\hat{\beta}_0$	17.29	50.35	16.17	50.92	30.89	49.98	31.46	50.78	29.14	48.20
$\hat{\beta}_1$	0.37	0.33	0.37	0.33	0.35	0.33	0.35	0.33	0.36	0.34
$\hat{\beta}_2$	−0.03	−0.13	−0.03	−0.13	−0.07	−0.13	−0.07	−0.13	−0.07	−0.13
$\hat{p}_e$	–	–	–	–	0.62	0.50	0.68	0.49	–	–
$\hat{\sigma}_v^2$	175.68	231.87	127.68	186.07	205.01	238.42	160.22	203.55	102.74	155.15
$\hat{\sigma}_1^2$	–	–	–	–	182.01	121.40	170.64	119.49	–	–
$\hat{\sigma}_2^2$	370.00	216.00	341.00	192.00	976.00	231.00	483.00	188.00	225.60	161.50

the posterior distribution for some of them may be very similar to the prior distribution. An overparameterized likelihood usually carries little information for some parameters responsible for overparameterization. In particular, if our mixture model is overparameterized in the sense that variances of mixture components are similar, then the integrated likelihood may be flat on the mixing proportion. We observe this scenario in our data analysis when we removed the suspected outlier observation from analysis based on our model. Since our mixture model is meant to accommodate outliers based on unequal variances for the mixing components, in the absence of any outliers the mixture of two normal distributions may not be required. In particular, we noticed earlier that with the suspected outlier removed the estimates of the two variance components σ_1^2 and σ_2^2 are very similar. Also, the posterior histogram of the mixing proportion p_e, not presented here, resembles a uniform distrubution, the prior distribution assigned in our Bayesian analysis. In fact, the posterior mean of this parameter for the reduced data is the same as the prior mean 0.5. This essentially says that the likelihood is devoid of any information about p_e to update the prior distribution.

One advantage of our mixture model is that it explicitly models any representative outlier through the latent indicator variable z_{ij}. By computing the posterior probability of $z_{ij} = 0$ we can compute the posterior probability that an observed y_{ij} is an outlier.

While the REBLUP method does not give a similar measure for an observation, one can determine the outlier status by computing the standardized residual associated with an observation. To show the effectiveness of our method, in Figure 1, we plotted the posterior probabilities of an individual observation being an outlier against the observation's standardized residual. In the left panel, we showed the plot of these posterior probabilities for the full data, and in the right panel we included the same by removing the suspected outlier. These two figures are in sharp contrast; the left panel clearly showed that there is a high probability (0.86) that the second observation in Hardin county is an outlier. The associated large negative standardized residual of this observation also confirmed that, and from this plot an approximate monotonicity of these posterior probabilities with respect to the absolute values of the standardized residuals may also be discerned. However, the right panel shows that for the reduced data excluding the suspected outlier, the standardized residuals for the remaining observations are between -3 and 3, with the associated posterior probabilities of being outlier observations are all between 0.44 and 0.64. None of these probabilities is particularly larger than prior probability 0.5 to indicate outlier status of that corresponding observation. This little change of the outlier prior probabilities in the posterior distribution for the reduced data essentially confirms that a discrete scale mixture of normal distributions is not supported by the data, or in other words, the scale mixture model is not required to explain the data, which is the same as that there are possibly no outliers in the data set.

6 A Simulation Study

In our extensive simulation study, we followed the simulation setup used by Sinha and Rao (2009). Corresponding to the model in (2.1), we use a single auxiliary variable x, which we generated independently from a normal distribution with mean 1 and variance 1. In our simulations we use $m = 40$. We generated 40 sets of 200 ($= N_i$) values of x to create the finite population of covariates for the 40 small areas. Based on these simulated values we computed $\bar{X}_i = \frac{1}{N_i}\sum_{j=1}^{N_i} x_{ij}$. Throughout our simulations we keep the generated x values

fixed. We used these generated x_{ij} values and generated $v_i, i = 1, \cdots, m$ independently from $N(0, \sigma_v^2)$ with $\sigma_v^2 = 1$. We generated $e_{ij}, j = 1, \cdots, N_i, i = 1, \cdots, m$ as iid from one of three possible distributions: (i) the case of no outliers where e_{ij} are generated from $N(0, 1)$ distribution; (ii) a mixture of normal distributions, with 10% outliers from a $N(0, 5^2)$ distribution and the remaining 90% from the $N(0, 1)$ distribution; and (iii) e_{ij}'s are iid from a t-distribution with 4 degrees of freedom. We also took $\beta_0 = 1$ and $\beta_1 = 1$ as in Sinha and Rao (2009), and generated m small area finite populations based on the generated x_{ij}'s, v_i's and e_{ij}'s by computing $Y_{ij} = \beta_0 + \beta_1 x_{ij} + v_i + e_{ij}$ based on the NER model in (2.1). Our goal is prediction of finite population small area means $\bar{Y}_i = \frac{1}{N_i} \sum_{j=1}^{N_i} Y_{ij}, i = 1, \cdots, m$. After examining no significant difference between $\bar{Y}_i$ and $\beta_0 + \beta_1 \bar{X}_i + v_i = \theta_i$ (say) in the simulated populations, as in Sinha and Rao (2009), we also consider prediction of θ_i.

From each simulated small area finite population we selected a simple random sample of size $n_i = 4$ for each small area. Based on the selected samples we derived the HB predictors of Datta and Ghosh (1991) (referred to as DG), the REBLUPs of Sinha and Rao (2009) (referred to as SR), the MQ predictors of Chambers et al. (2014) (referred to as CCST-MQ, based on their equation (38)) and our proposed robust HB predictors (referred to as NM). In addition to the point predictors we also obtained the posterior variances of both the HB predictors and the estimates of the MSE of the REBLUPs based on the bootstrap method proposed by Sinha and Rao (2009), and the estimates of MSE of the MQ predictors, obtained by using pseudo-linearization in equation (39) of Chambers et al. (2014).

For each simulation setup, we have simulated $S = 100$ populations. For the sth created population, $s = 1, \cdots, S$, we computed the values of $\theta_i^{(s)}$, which will be treated as the true values. We denote the sth simulation sample by $d^{(s)}$, and based on this data we calculate the REBLUP predictors $\hat{\theta}_{i,SR}^{(s)}$ and their estimated MSE, $mse(\hat{\theta}_{i,SR}^{(s)})$ using the procedure proposed by Sinha and Rao (2009). To assess the accuracy of the point predictors we computed the empirical bias $eB_{i,SR} = \frac{1}{S} \sum_{s=1}^{S} (\hat{\theta}_{i,SR}^{(s)} - \theta_i^{(s)})$ and empirical MSE

$eM_{i,SR} = \frac{1}{S}\sum_{s=1}^{S}(\hat{\theta}_{i,SR}^{(s)} - \theta_i^{(s)})^2$. Treating $eM_{i,SR}$ as the "true" measure of variability of $\hat{\theta}_{i,SR}$, we also evaluate the accuracy of the MSE estimator $mse(\hat{\theta}_{i,SR})$, suggested by Sinha and Rao (2009). Accuracy of the MSE estimator is evaluated by the relative difference between the empirical MSE and the average (over simulations) estimated MSE, given by $RE_{mse-SR,i} = \{(1/S)\sum_{s=1}^{S} mse(\hat{\theta}_{i,SR}^{(s)}) - eM_{i,SR}\}/eM_{i,SR}$. Similarly, we obtained the predictors $\hat{\theta}_{i,CCST}^{(s)}$, estimated MSEs $mse(\hat{\theta}_{i,CCST}^{(s)})$ of Chambers et al. (2014), empirical biases and empirical MSEs of point estimators and relative biases of the estimated MSEs. Using the point estimates and MSE estimates we created approximate 90% prediction intervals $I_{i,SR,90}^{(s)} = [\hat{\theta}_{i,SR}^{(s)} - 1.645\sqrt{mse(\hat{\theta}_{i,SR}^{(s)})}, \hat{\theta}_{i,SR}^{(s)} + 1.645\sqrt{mse(\hat{\theta}_{i,SR}^{(s)})}]$ and 95% prediction intervals $I_{i,SR,95}^{(s)} = [\hat{\theta}_{i,SR}^{(s)} - 1.96\sqrt{mse(\hat{\theta}_{i,SR}^{(s)})}, \hat{\theta}_{i,SR}^{(s)} + 1.96\sqrt{mse(\hat{\theta}_{i,SR}^{(s)})}]$. We also obtained similar intervals for the MQ method of Chambers et al. (2014). We evaluated empirical biases, empirical MSEs, relative biases of estimated MSEs, and empirical coverage probabilities of prediction intervals for all four methods. These quantities for all 40 small areas are plotted in Figures 2, 3 and 4.

We plotted the empirical biases on the left panel and the empirical MSEs on the right panel of Figure 2. These estimators do not show any systematic bias. In terms of eM, the REBLUP and the proposed NM HB predictor appear to be most accurate and perform similarly (in fact, based on all evaluation criteria considered here, the proposed NM HB and the REBLUP methods have equivalent performance). In terms of eM, the MQ predictor has maximum variability and the standard DG HB predictor is in third place. In the case of no outliers, while the other three predictors have the same eM, the MQ predictor is slightly more variable. Moreover, we examined how closely the posterior variances of the Bayesian predictors and the MSE estimators of the frequentist robust predictors track their respective eM of prediction (see Figure 3). The posterior variance of the proposed NM HB predictor and the estimated MSE of REBLUP appear to track the eM the best without any evidence of bias. The posterior variance of the standard HB predictor appears to overestimate the eM and the estimated MSE of the MQ predictor appears to underestimate. An undesirable consequence of this negative bias of the MSE estimator of the MQ method is that the related prediction intervals often fail to cover

the true small area means (see the plots in Figure 4).

Our sampling-based Bayesian approach allowed us to create credible intervals for the small area means at the nominal levels of 0.90 and 0.95 based on sample quantiles of the Gibbs samples of the θ_i's. For the Sinha-Rao and the Chambers et al. methods we used their respective estimated root MSE of the REBLUPs or MQ-predictors to create symmetric approximate 90% and 95% prediction intervals of the small area means.

To assess the coverage rate of these prediction intervals we computed empirical coverage probabilities $eC_{i,SR,90} = \frac{1}{S}\sum_{s=1}^{S} I[\theta_i^{(s)} \in I_{i,SR,90}^{(s)}]$ and $eC_{i,SR,95} = \frac{1}{S}\sum_{s=1}^{S} I[\theta_i^{(s)} \in I_{i,SR,95}^{(s)}]$, where $I[x \in A]$ is the usual indicator function that is one for $x \in A$ and 0 otherwise.

Based on the same setup and same set of simulated data we also evaluated the two HB procedures. In the Bayesian approach, the point predictor, the posterior variance and the credible intervals for $\theta_i^{(s)}$ in the sth simulation were computed based on the MCMC samples of $\theta_i^{(s)}$ from its posterior distribution, generated by Gibbs sampling. The posterior mean and posterior variance are computed by the sample mean and the sample variance of the MCMC samples. An equi-tailed $100(1-2\alpha)\%$ credible interval for $\theta_i^{(s)}$ is created, where the lower limit is the 100αth sample percentile and the upper limit is the $100(1-\alpha)$th sample percentile of the MCMC samples of $\theta_i^{(s)}$ from the sth simulation.

Suppose in the sth simulation $\hat{\theta}_{i,DG}^{(s)}$ denotes the Datta-Ghosh HB predictor of θ_i and $V_{i,DG}^{(s)}$ denotes the posterior variance. The empirical bias of the Datta-Ghosh predictor of θ_i is defined by $eB_{i,DG} = \frac{1}{S}\sum_{s=1}^{S}(\hat{\theta}_{i,DG}^{(s)} - \theta_i^{(s)})$ and empirical MSE by $eM_{i,DG} = \frac{1}{S}\sum_{s=1}^{S}(\hat{\theta}_{i,DG}^{(s)} - \theta_i^{(s)})^2$. To investigate the extent $V_{i,DG}^{(s)}$ may be interpreted as an estimated mse of the predictor $\hat{\theta}_{i,DG}$, we compute the relative difference between the empirical MSE and the average (over simulations) posterior variance, given by $RE_{V-DG,i} = \{(1/S)\sum_{s=1}^{S} V_{i,DG}^{(s)} - eM_{i,DG}\}/eM_{i,DG}$. These quantities for all 40 small areas are plotted in Figure 3.

Based on the MCMC samples of θ_i's for the sth simulated data set, let $I_{i,DG,90}^{(s)}$ be the 90% credible interval for θ_i. To evaluate the frequentist coverage probability of the credible

interval for θ_i we computed empirical coverage probabilities $eC_{i,DG,90} = \frac{1}{S}\sum_{s=1}^{S} I[\theta_i^{(s)} \in I_{i,DG,90}^{(s)}]$. Corresponding to a credible interval $I_{i,DG,90}^{(s)}$, we use $L_{i,DG,90}^{(s)}$ to denote its length, and computed empirical average length of a 90% credible interval for θ_i based on Datta-Ghosh approach by $\bar{L}_{i,DG,90} = \frac{1}{S}\sum_{s=1}^{S} L_{i,DG,90}^{(s)}$. Similarly, we computed $eC_{i,DG,95}$ and $\bar{L}_{i,DG,95}$ for the 95% credible intervals for θ_i.

Finally, as we did for the Datta-Ghosh HB predictor, we computed similar quantities for our new robust HB predictor. Specifically, suppose $\hat{\theta}_{i,NM}^{(s)}$ is the newly proposed NM HB predictor of $\theta_i^{(s)}$ and $V_{i,NM}^{(s)}$ is the posterior variance. For the new predictor we define the empirical bias by $eB_{i,NM} = \frac{1}{S}\sum_{s=1}^{S}(\hat{\theta}_{i,NM}^{(s)} - \theta_i^{(s)})$ and empirical MSE by $eM_{i,NM} = \frac{1}{S}\sum_{s=1}^{S}(\hat{\theta}_{i,NM}^{(s)} - \theta_i^{(s)})^2$. Again, to investigate the extent $V_{i,NM}^{(s)}$ may be viewed as an estimated MSE of the predictor $\hat{\theta}_{i,NM}$, we computed the relative difference between the emprical MSE and the average (over simulations) posterior variance, given by $RE_{V-NM,i} = \{(1/S)\sum_{s=1}^{S} V_{i,NM}^{(s)} - eM_{i,NM}\}/eM_{i,NM}$. These quantities for all 40 small areas are plotted in Figure 3. Based on the MCMC samples of θ_i's for the sth simulated data set, let $I_{i,NM,90}^{(s)}$ be the 90% credible interval for θ_i. To evaluate the frequentist coverage probability of the credible interval for θ_i we computed empirical coverage probabilities $eC_{i,NM,90} = \frac{1}{S}\sum_{s=1}^{S} I[\theta_i^{(s)} \in I_{i,NM,90}^{(s)}]$. Corresponding to a credible interval $I_{i,NM,90}^{(s)}$, we use $L_{i,NM,90}^{(s)}$ to denote its length, and computed empirical average length of a 90% credible interval for θ_i based on new approach by $\bar{L}_{i,NM,90} = \frac{1}{S}\sum_{s=1}^{S} L_{i,NM,90}^{(s)}$. Similarly, we computed $eC_{i,NM,95}$ and $\bar{L}_{i,NM,95}$ for the 95% credible intervals for θ_i.

We plotted the empirical coverage probabilities for the four methods that we considered in this article. The plot reveals significant undercoverage of the approximate prediction intervals created by using the estimated prediction MSE proposed by Chambers et al. (2014). This undercoverage is not surprising since their estimated MSE mostly underestimates the true MSE (measured by the eM) (see Figure 3). Coverage probabilities of the Sinha-Rao prediction intervals and the two Bayesian credible intervals are remarkably accurate. This lends dual interpretation of our proposed credible intervals, Bayesian by construction, and frequentist by simulation validation. This property is highly desira-

ble to practitioners, who often do not care about a paradigm or a philosophy. In the same plot, we also plotted the ratio of the average lengths of the DG credible intervals to the newly proposed robust HB credible intervals. These plots show the superiority of the proposed method, yielding intervals which meet coverage accurately with average lengths about 25-30% shorter compared to the DG method for normal mixture model with 10% contamination. Again these two intervals meet the coverage accurately when the unit-level errors are generated from normal (no outliers) or a moderately heavy-tail distribution (t_4). In these cases, the reduction in length of the intervals is less, which is about 10%. This shorter prediction intervals from the new method even for normal distribution for the unit-level error is interesting; it shows that the proposed method does not lose any efficiency in comparison with the Datta-Ghosh method even when the normality of the unit-level errors holds.

The comparison of NM HB prediction intervals and the Sinha-Rao prediction intervals yields a mixed picture. In the mixture setup, the NM HB prediction intervals attained coverage probability more accurately than the Sinha-Rao intervals, which undercover by 1%, and on an average the Bayesian prediction intervals are about 2% shorter than the frequentist intervals. When the data are simulated from a t_4 distribution, the coverage probabilities of the Sinha-Rao prediction intervals are about 1% below the target, but these intervals are about 3% shorter than the NM HB prediction intervals, which attained the nominal coverage. Finally, when the population does not include any outlier, these two methods perform the same, both attained the nominal coverage and yield the same average length.

7 Conclusion

The NER model by Battese et al. (1988) plays an important role in small area estimation for unit-level data. While Battese et al. (1988), Prasad and Rao (1990) and Datta and Lahiri (2000) investigated EBLUPs of small area means, Datta and Ghosh (1991) proposed an HB approach for this model. Sinha and Rao (2009) investigated robustness

of the MSE estimates of EBLUPs in Prasad and Rao (1990) for outliers in the response. They showed in presence of outliers robustness of their REBLUPs and lack of robustness of the EBLUPs.

In this article we showed that non-robustness also persists for the HB predictors by Datta and Ghosh (1991). To deal with this undesirable issue we proposed an alternative to the HB predictors by using a mixture of normal distributions for the unit-level error part of the NER model. An illustrative application and simulation study show the superiority of our proposed method over the existing HB, EBLUP and M-quantile solutions. Indeed simulation results show the superiority of our method over the Datta and Ghosh (1991) HB predictors and the M-quantile small area estimators of Chambers et al. (2014). Performance of our proposed NM HB method is found to be as good as the frequentist solution of Sinha and Rao (2009). Our proposed Bayesian intervals also achieve the corresponding frequentist coverage. Thus, unlike the frequentist solutions, our proposed HB solution enjoys dual interpretation, Bayesian by construction, and frequentist via simulation, a feature attractive to practitioners. Moreover, suggested credible intervals are shorter in length in comparison with the other nominal prediction intervals. In fact, the application and simulations show the proposed NM HB method is the best among the four methods in presence of outliers. Our proposed method is as good as the HB method of Datta and Ghosh (1991), even in absence of outliers. Thus there will be no loss in using the proposed HB method for all data sets. It is not clear to us that why M-quantile performs poorly. However, we note that in our simulations, all the errors are centered at zero. Alternatively, one can explore the performance of these methods when the outlier parts of the respective error components are generated from a distribution which is not centered at zero. This remains a topic of future research.

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Supporting Information

Property of the posterior distribution corresponding to the proposed model has been discussed in the supplementary material.

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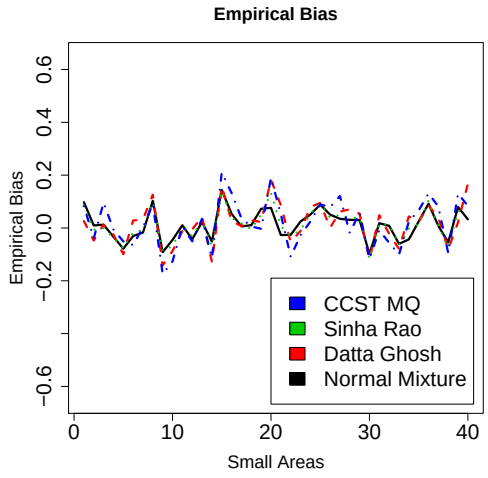

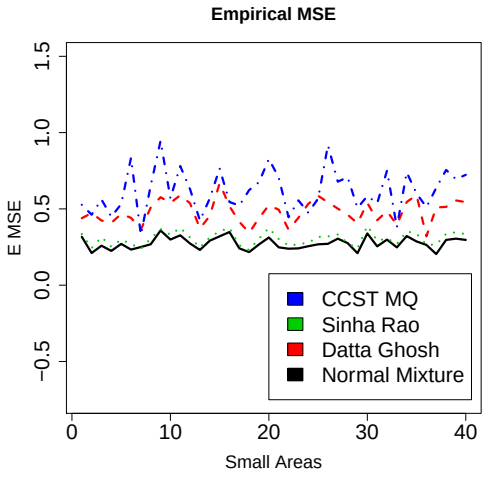

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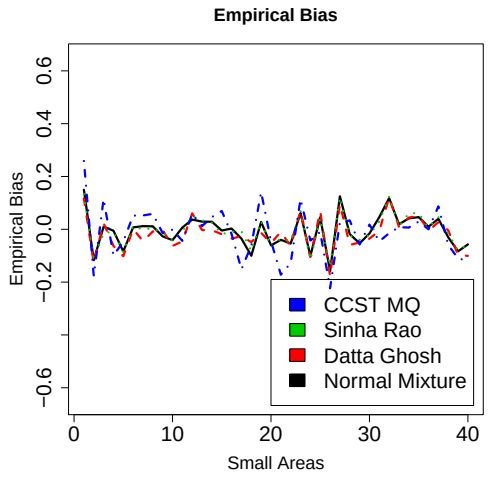

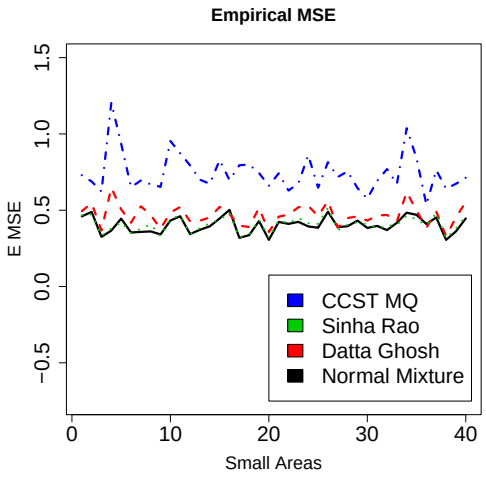

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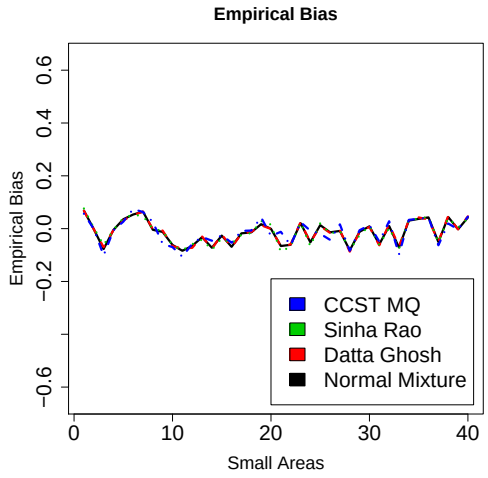

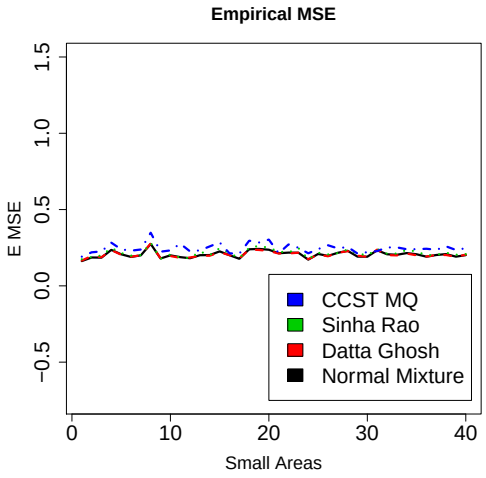

Figure 2: Plot of empirical biases and empirical MSEs of $\hat{\theta}$s

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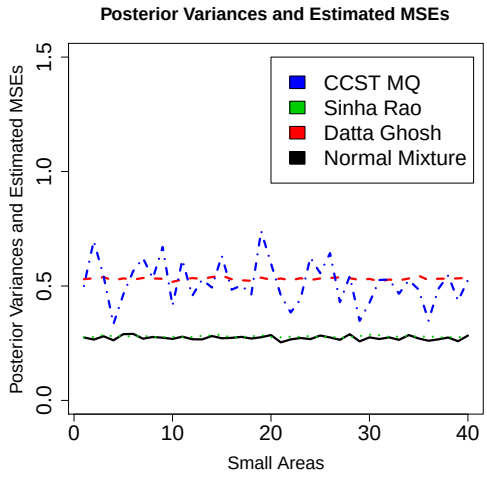

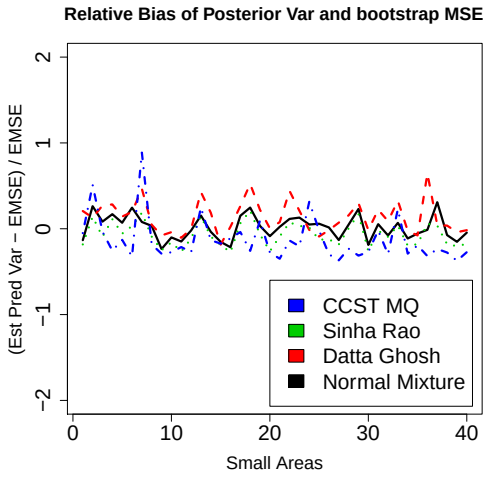

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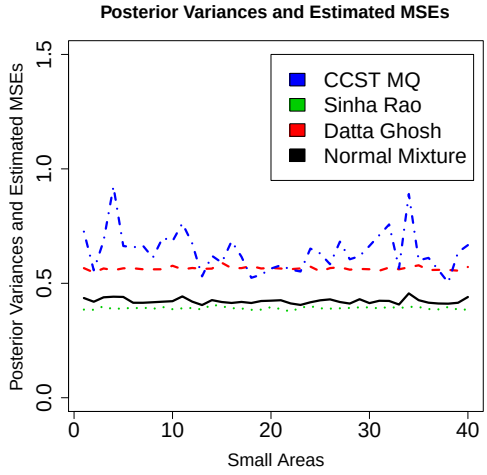

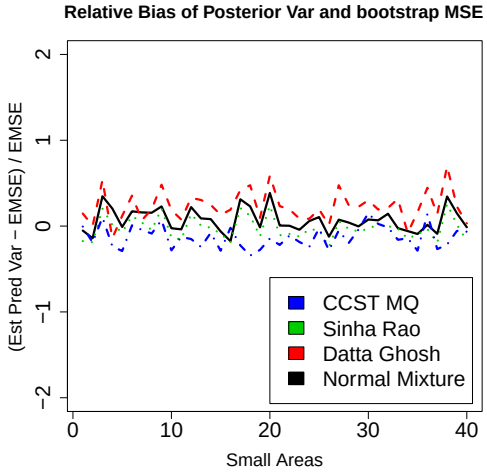

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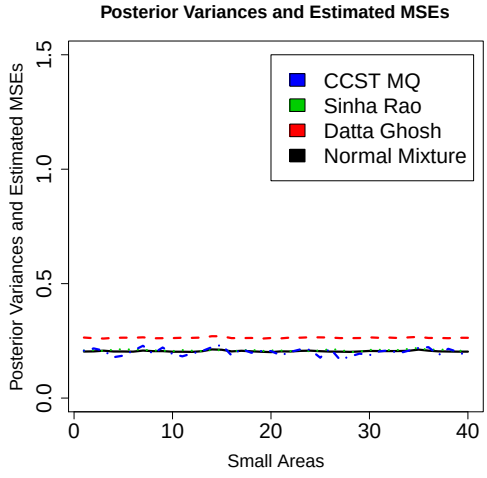

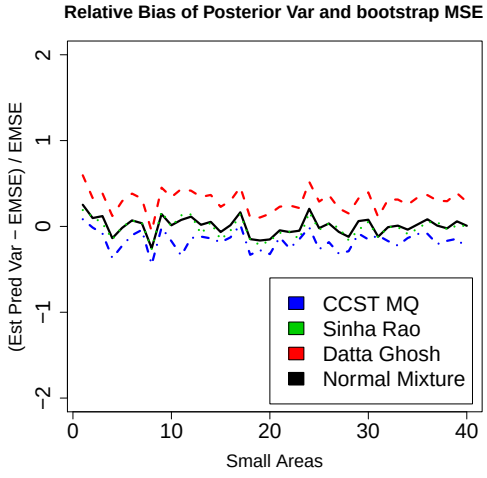

Figure 3: Plot of posterior variances and MSE estimates and their empirical relative biases

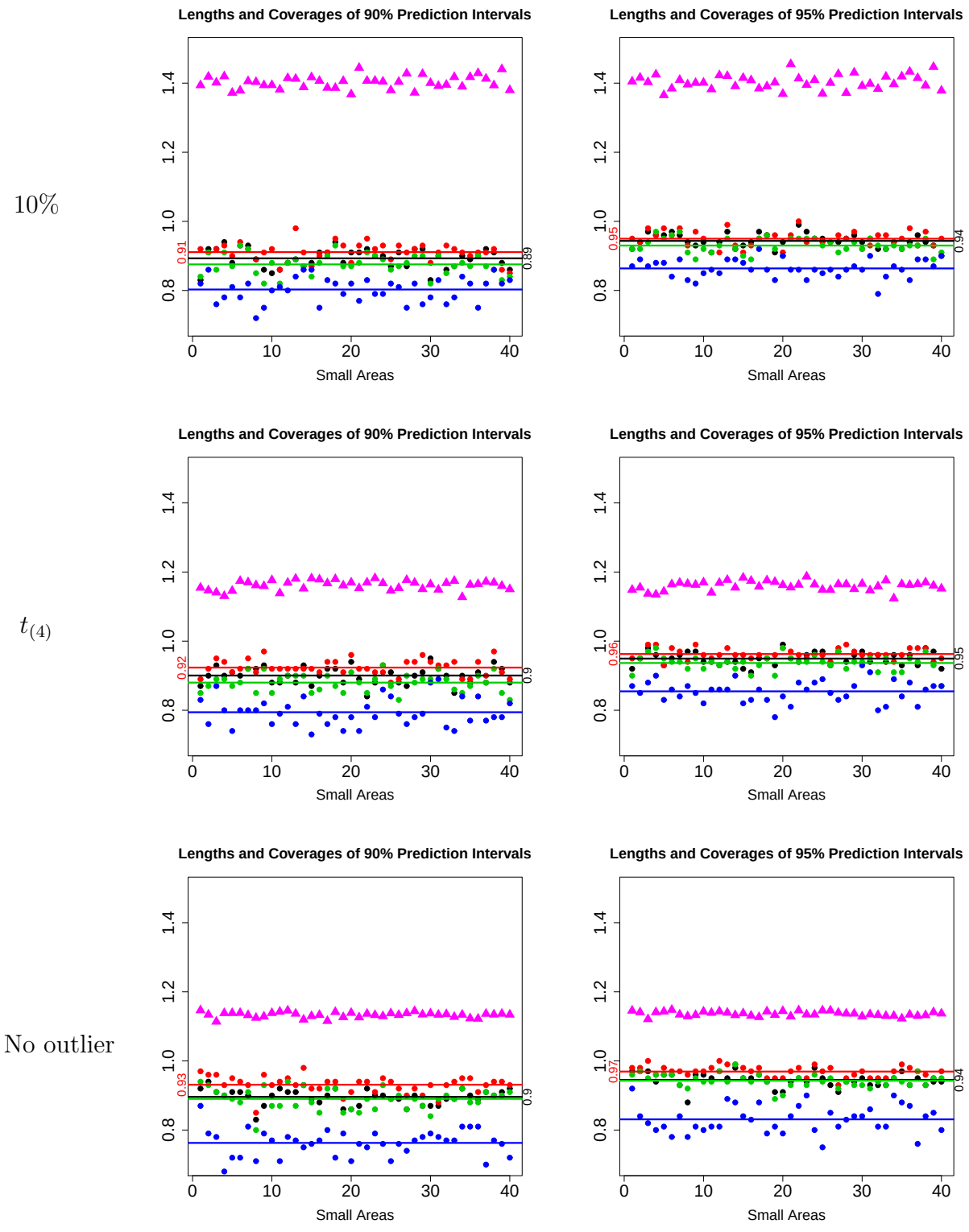

Figure 4: Plot of lengths and coverages of credible and prediction intervals

Robust Hierarchical Bayes Small Area Estimation for the Nested Error Linear Regression Model

8 Supplementary Materials

8.1 Exploration of the Propriety of the Posterior Density

Since improper prior distribution has been used in the HB model proposed in this article, it is important to explore the propriety of the resulting posterior distribution in order to avoid misleading results based on improper posteriors (cf. Hobert and Casella, 1996). In the following results, we first provide sufficient conditions for the propriety of the resulting posterior distribution based on the proposed model. We relax the condition $n_i \geq 2$ for all areas in the corollary below.

Theorem 8.1 *Let $\sum_{i=1}^{m} n_i = n$. The following conditions are sufficient for the propriety of the posterior distribution under the proposed model:*

(a) $n_i \geq 2$ *for* $i = 1, \ldots, m$,

(b) $n \geq 2m + 2p - 1$,

(c) $m \geq p + 6$.

A detailed proof of Theorem 8.1 is provided in Section 8.2 of the Supplementary Materials. While Theorem 8.1 appears to be restrictive, the following corollary and lemma show that it is not the case.

Corollary 8.2 *If there exists a set S of m' $(m' \leq m)$ small areas such that*

(a) $n_i \geq 2$, n_i *being the number of sampled units from the* i^{th} *small area,* $i \in S$,

(b) $\sum_{i \in S} n_i \geq 2m' + 2p - 1$,

(c) $m' \geq p + 6$,

then the posterior distribution under the proposed model will be proper.

Proof of Corollary 8.2: Proof follows from an Application of the lemma below.

Lemma 8.3 *Let $\theta \sim \pi(\theta)$ and $d|\theta \sim f(d|\theta)$. We partition d as $d = (d^{(1)T}, d^{(2)T})^T$. If the posterior distribution $\theta|d^{(1)}$ is proper, then the posterior distribution $\theta|d$ is also proper.*

Suppose there exists $m' (\leq m)$ small areas which satisfy conditions (a), (b) and (c) of Theorem 8.1. Let $S_{m'}$ be the set of small areas with at least two sampled units and $S^c_{m'}$ contain rest of the small areas. Let us partition the responses for the sampled units as follows:

$$d^{(1)} = \{y_{ij} : i \in S_{m'}; j = 1, \ldots, n_i\} \text{ and } d^{(2)} = \{y_{ij} : i \in S^c_{m'}; j = 1, \ldots, n_i\}.$$

Let θ be the set of model parameters. By Theorem 8.1, $f(\theta|d^{(1)})$ is proper. Now, applying Lemma 8.3, we can say $f(\theta|d) = f(\theta|d^{(1)}, d^{(2)})$ is proper. This proves Corollary 8.2.

8.2 Proof of the Theorem

Proof of Theorem 8.1: We assume that there are at least two sampled units for each small area, i.e. $n_i \geq 2$, $i = 1, \ldots, m$; and $n \geq 2m + 2p - 1$, where $n = \sum_{i=1}^m n_i$. At first, we consider the case when $n = 2m + 2p - 1$, the argument can be extended to the case $n > 2m + 2p - 1$ by applying Lemma 8.3. Under the proposed model, the joint pdf of y_{ij}'s, $j = 1, \ldots, n_i$, $i = 1, \ldots, m$; $v\,(m \times 1)$, $\beta\,(p \times 1)$, σ_1^2, σ_2^2, σ_v^2 and p_e is given by

$$\begin{aligned} f(y, v, \beta, \sigma_2^2, \sigma_1^2, \sigma_v^2, p_e) \propto \quad & \sum_{\Omega} \left[\prod_{i=1}^{m} \left\{ \prod_{k=1}^{n_{i1}} \frac{p_e}{\sqrt{\sigma_1^2}} \exp\left(-\frac{1}{2} \frac{(y_{ij_k} - x_{ij_k}^T \beta - v_i)^2}{\sigma_1^2} \right) \right\} \right. \\ & \left. \times \left\{ \prod_{k=n_{i1}+1}^{n_i} \frac{(1-p_e)}{\sqrt{\sigma_2^2}} \exp\left(-\frac{1}{2} \frac{(y_{ij_k} - x_{ij_k}^T \beta - v_i)^2}{\sigma_2^2} \right) \right\} \right] \\ & \times \frac{1}{(\sigma_v^2)^{\frac{m}{2}}} \exp\left(-\frac{1}{2} \sum_{i=1}^{m} \frac{v_i^2}{\sigma_v^2} \right) \times \frac{1}{(\sigma_2^2)^2} I(\sigma_1^2 < \sigma_2^2) \quad (8.1) \end{aligned}$$

The summation $\sum_{\Omega}$ and the quantities n_{i1}, n_{i2}, $i = 1, \ldots, m$ are explained below. Let $z_{ij} = 1$, if the j^{th} sampled unit of the i^{th} small area corresponds to the mixture component σ_1^2 and $z_{ij} = 0$ otherwise. The set Ω contains all possible choices of $\mathbf{z} = (z_{11}, \ldots, z_{mn_m})$ vector. Hence the cardinality of Ω is 2^n. For a given $\mathbf{z}$, let $n_{i1} = \sum_{j=1}^{n_i} z_{ij}$ and $n_{i2} = n_i - n_{i1}$ for $i = 1, \ldots, m$. Then n_{i1} is the number of units from the i^{th} small area whose unit-level

variance corresponds to the mixture component σ_1^2. The remaining n_{i2} units from the i^{th} small area corresponds to the mixture component σ_2^2.

Define, $S_1 = \{i : n_{i1} > 0\}$ and $S_2 = \{i : n_{i2} > 0\}$. Clearly, $S_1 \cup S_2 = \{1, \ldots, m\}$ and $S_1 \cap S_2$ may not be an empty set. Let m_i be the cardinality of S_i, $i = 1, 2$, then $m \leq m_1 + m_2$. Note that n_{i1} or n_{i2} can be zero for some i. Indeed, if $i \notin S_1, n_{i1} = 0$ and if $i \notin S_2, n_{i2} = 0$. Define, $n_1^* = \sum_{i \in S_1} n_{i1}$ and $n_2^* = \sum_{i \in S_2} n_{i2}$.

From (8.1), a typical term under the sum over Ω is,

$$\begin{aligned}
&\varphi(y, v, \beta, \sigma_1^2, \sigma_2^2, \sigma_v^2, p_e) \\
&\quad = C \times p_e^{n_1^*}(1 - p_e)^{n_2^*} \times \frac{1}{(\sigma_1^2)^{\frac{n_1^*}{2}}} \times \exp\left(-\frac{1}{2\sigma_1^2} \sum_{i \in S_1} \sum_{k=1}^{n_{i1}} (y_{ij_k} - x_{ij_k}^T \beta - v_i)^2\right) \\
&\qquad \times \frac{1}{(\sigma_2^2)^{\frac{n_2^*}{2}}} \times \exp\left(-\frac{1}{2\sigma_2^2} \sum_{i \in S_2} \sum_{k=n_{i1}+1}^{n_i} (y_{ij_k} - x_{ij_k}^T \beta - v_i)^2\right) \\
&\qquad \times \frac{1}{(\sigma_v^2)^{\frac{m}{2}}} \exp\left(-\frac{1}{2} \sum_{i=1}^{m} \frac{v_i^2}{\sigma_v^2}\right) \times \frac{I(\sigma_1^2 < \sigma_2^2)}{(\sigma_2^2)^2}, \qquad (8.2)
\end{aligned}$$

where C is a generic, positive constant. In order to check the integrability of $f(y, v, \beta, \sigma_2^2, \sigma_1^2, \sigma_v^2, p_e)$ with respect to $\beta,\ v,\ \sigma_1^2,\ \sigma_2^2,\ \sigma_v^2,\ p_e$ in (8.1), we need to check the integrability of each typical term in (8.1) with respect to $\beta,\ v,\ \sigma_1^2,\ \sigma_2^2,\ \sigma_v^2,\ p_e$.

We introduce the following notation: $y_1 = \text{col}_{i \in S_1} \text{col}_{1 \leq k \leq n_{i1}} y_{ij_k}$; $X_1 = \text{col}_{i \in S_1} \text{col}_{1 \leq k \leq n_{i1}} x_{ij_k}^T$ and $y_2 = \text{col}_{i \in S_2} \text{col}_{n_{i1}+1 \leq k \leq n_i} y_{ij_k}$; $X_2 = \text{col}_{i \in S_2} \text{col}_{n_{i1}+1 \leq k \leq n_i} x_{ij_k}^T$, $Z_1 = \bigoplus_{i=1}^{m} 1_{n_{i1}}$ and $Z_2 = \bigoplus_{i=1}^{m} 1_{n_{i2}}$.

Note that, there are m_1 and m_2 components of v are involved in $Z_1 v$ and $Z_2 v$ respectively. Let the rank of X_1 $(n_1^* \times p)$ and X_2 $(n_2^* \times p)$ be p_1 and p_2 respectively, where $p_1 + p_2 \geq p$. We now state the lemma below.

Lemma 8.4 *If $n = 2m + 2p - 1$ and $m \geq p + 6$, then one of the following conditions must hold. (a) $n_1^* \geq m_1 + p_1$, $m_1 > 3$ or (b) $n_2^* \geq m_2 + p_2$, $m_2 > 3$.*

The proof of Lemma 8.4 is provided in Section 8.3. Without loss of generality, for the rest of the proof, we assume that $n_1^* \geq m_1 + p_1$ and $m_1 > 3$. Had we assumed $n_2^* > m_2 + p_2$, $m_2 > 3$, it will lead us to establish the same results. Note that we do not have to make

separate assumptions for n_1^* and m_1, they come from the assumptions $n \geq 2m + 2p - 1$ and $m \geq p + 6$.

Without loss of generality, we assume that the rows of X_1 are arranged such that the first p_1 rows are linearly independent. These rows constitute a submatrix $X_{11}(p_1 \times p)$, the rest of the rows of X_1 can be expressed as $A_{21}X_{11}$ for some matrix $A_{21}((n_1^* - p_1) \times p_1)$. Similarly, we assume that first p_2 rows of X_2 are so arranged that they are linearly independent. Since, $p_2 \geq p - p_1$, we further assume that the first $(p - p_1)$ of these p_2 rows are linearly independent of the rows of X_{11}, we denote this portion of X_2 as the sub matrix $X_{211}((p - p_1) \times p)$.

Let, X_{212} consists next $p_2 - (p - p_1)$ linearly independent rows of X_2, and X_{22} contains the remaining $(n_2^* - p_2)$ rows. Hence,

$$X_1 = \begin{pmatrix} X_{11} \\ A_{21}X_{11} \end{pmatrix} \qquad \text{and} \qquad X_2 = \begin{pmatrix} X_{211} \\ X_{212} \\ X_{22} \end{pmatrix}$$

where, $\text{rank}(X_{11}) = p_1$.

According to the construction of the matrices, $X_{212} = H \begin{pmatrix} X_{11} \\ X_{211} \end{pmatrix}$ for some $H = \begin{pmatrix} H_{21} & H_{22} \end{pmatrix}$, note that $H_{21} \neq 0$. we can write, $X_{22} = A_{22} \begin{pmatrix} X_{211} \\ X_{212} \end{pmatrix}$ for some $A_{22}((n_2^* - p_2) \times p_2)$.

We consider the transformation: $\rho_1 = X_{11}\beta$ and $\rho_2 = X_{211}\beta$; $\rho = (\rho_1^T, \rho_2^T)^T$.

Now, $X_1\beta = \begin{pmatrix} X_{11} \\ A_{21}X_{11} \end{pmatrix} \beta = \begin{pmatrix} I_{p_1} \\ A_{21} \end{pmatrix} X_{11}\beta = M_1\rho_1$, where $M_1 = \begin{pmatrix} I_{p_1} \\ A_{21} \end{pmatrix}$, $\text{rank}(M_1) = p_1$.

Similarly,

$$X_{212}\beta = H \begin{pmatrix} X_{11}\beta \\ X_{211}\beta \end{pmatrix} = H \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = H\rho \text{ and}$$

$$X_{22}\beta = A_{22} \begin{pmatrix} X_{211} \\ X_{212} \end{pmatrix} \beta = A_{22} \begin{pmatrix} \rho_2 \\ H\rho \end{pmatrix} = A_{22} \begin{pmatrix} 0 & I \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = A_{22}^*\rho,$$

hence, $X_2\beta = \begin{pmatrix} X_{211} \\ X_{212} \\ X_{22} \end{pmatrix} \beta = \begin{pmatrix} \rho_2 \\ H\rho \\ A_{22}^*\rho \end{pmatrix} = \begin{pmatrix} \rho_2 \\ G\rho \end{pmatrix}$, where $G = \begin{pmatrix} H \\ A_{22}^* \end{pmatrix}$, we partition y_2 and Z_2 according to the partitioned rows of X_2, i.e., $y_2 = \begin{pmatrix} y_{211} \\ y_{212} \\ y_{22} \end{pmatrix} = \begin{pmatrix} y_{211} \\ y_2^* \end{pmatrix}$, where $y_2^* = \begin{pmatrix} y_{212} \\ y_{22} \end{pmatrix}$ and $Z_2 = \begin{pmatrix} Z_{211} \\ Z_{212} \\ Z_{22} \end{pmatrix} = \begin{pmatrix} Z_{211} \\ Z_{22}^* \end{pmatrix}$, where $Z_{22}^* = \begin{pmatrix} Z_{212} \\ Z_{22} \end{pmatrix}$.

After these transformations, we can rewrite the right hand side of (8.2) as

$$
\begin{aligned}
&= C \times \frac{1}{(\sigma_1^2)^{\frac{n_1^*}{2}}} \exp\left(-\frac{1}{2\sigma_1^2}\left(y_1 - M_1\rho_1 - Z_1 v\right)^T \left(y_1 - M_1\rho_1 - Z_1 v\right)\right) \\
&\quad \times \frac{1}{(\sigma_2^2)^{\frac{p-p_1}{2}}} \exp\left(-\frac{1}{2\sigma_2^2}\left(y_{211} - \rho_2 - Z_{211} v\right)^T \left(y_{211} - \rho_2 - Z_{211} v\right)\right) \\
&\quad \times \frac{1}{(\sigma_2^2)^{\frac{n_2^* - (p-p_1)}{2}}} \exp\left(-\frac{1}{2\sigma_2^2}\left(y_2^* - G\rho - Z_2^* v\right)^T \left(y_2^* - G\rho - Z_2^* v\right)\right) \\
&\quad \times \frac{1}{(\sigma_v^2)^{\frac{m}{2}}} \exp\left(-\frac{v^T v}{2\sigma_v^2}\right) \times \frac{I(\sigma_1^2 < \sigma_2^2)}{(\sigma_2^2)^2} \\
&= \tilde{\varphi}(y_1, y_{211}, y_2^*, v, \rho_1, \rho_2, \sigma_1^2, \sigma_2^2, \sigma_v^2, p_e). \qquad (8.3)
\end{aligned}
$$

We integrate with respect to y_2^*, ρ_2 and σ_2^2 respectively, to obtain

$$
\begin{aligned}
&\int \tilde{\varphi}(y_1, y_{211}, y_2^*, v, \rho_1, \rho_2, \sigma_1^2, \sigma_2^2, \sigma_v^2, p_e)\, \mathrm{d}y_2^*\, \mathrm{d}\rho_2\, \mathrm{d}\sigma_2^2 \\
&\qquad = C \times \frac{1}{(\sigma_1^2)^{\frac{n_1^*+2}{2}}} \exp\left(-\frac{1}{2\sigma_1^2}\left(y_1 - M_1\rho_1 - Z_1 v\right)^T \left(y_1 - M_1\rho_1 - Z_1 v\right)\right) \\
&\qquad\quad \times \frac{1}{(\sigma_v^2)^{\frac{m}{2}}} \exp\left(-\frac{v^T v}{2\sigma_v^2}\right). \qquad (8.4)
\end{aligned}
$$

As we mentioned earlier, there are m_1 components of v involved in $Z_1 v$. We write those m_1 components as $v^{(1)} = (v_{i_1}, \ldots, v_{i_{m_1}})^T$. Then, $Z_1 v$ reduces to $Z_1^{(1)} v^{(1)}$, where $Z_1^{(1)} = \bigoplus_{j=1}^{m_1} 1_{n_{i_j 1}}$. Clearly, $\mathrm{rank}(Z_1^{(1)}) = m_1$ and $n_1^* = \sum_{j=1}^{m_1} n_{i_j 1}$. We integrate out $v^{(2)} = \{v_l :$

$l \in S \setminus S_1\}$,

$$\begin{aligned}
&\int \tilde{\varphi}(y_1, y_{211}, y_2^*, v, \rho_1, \rho_2, \sigma_1^2, \sigma_2^2, \sigma_v^2, p_e)\, \mathrm{d}y_2^*\, \mathrm{d}\rho_2\, \mathrm{d}\sigma_2^2\, \mathrm{d}v^{(2)} \\
&\quad = C \times \frac{1}{(\sigma_1^2)^{\frac{n_1^*+2}{2}}} \exp\left(-\frac{1}{2\sigma_1^2}\left(y_1 - M_1\rho_1 - Z_1^{(1)}v^{(1)}\right)^T\left(y_1 - M_1\rho_1 - Z_1^{(1)}v^{(1)}\right)\right) \\
&\qquad \times \frac{1}{(\sigma_v^2)^{\frac{m_1}{2}}} \exp\left(-\frac{v^{(1)T}v^{(1)}}{2\sigma_v^2}\right). \\
&\quad = C \times \frac{1}{(\sigma_1^2)^{\frac{n_1^*+2}{2}}} \exp\left\{-\frac{y_1^{*^T}\left\{I - M_1(M_1^TM_1)^{-1}M_1^T\right\}y_1^*}{2\sigma_1^2}\right\} \\
&\qquad \times \exp\left(-\frac{(\rho_1 - \hat{\rho}_1)^T(M_1^TM_1)^{-1}(\rho_1 - \hat{\rho}_1)}{2\sigma_1^2}\right) \times \frac{1}{(\sigma_v^2)^{\frac{m_1}{2}}} \exp\left(-\frac{v^{(1)T}v^{(1)}}{2\sigma_v^2}\right), \qquad (8.5)
\end{aligned}$$

where $y_1^* = y_1 - Z_1^{(1)}v^{(1)}$ and $\hat{\rho}_1 = (M_1^TM_1)^{-1}M_1^Ty_1^*$.

We integrate with respect to ρ_1 to get

$$\begin{aligned}
&\int \tilde{\varphi}(y_1, y_{211}, y_2^*, v, \rho_1, \rho_2, \sigma_1^2, \sigma_2^2, \sigma_v^2, p_e)\, \mathrm{d}y_2^*\, \mathrm{d}\rho_2\, \mathrm{d}\sigma_2^2\, \mathrm{d}v^{(2)}\mathrm{d}\rho_1 \\
&\quad = C \times \frac{1}{(\sigma_1^2)^{\frac{n_1^*-p_1+2}{2}}} \exp\left(-\frac{(y_1 - Z_1^{(1)}v^{(1)})^TR_1(y_1 - Z_1^{(1)}v^{(1)})}{2\sigma_1^2}\right) \\
&\qquad \times \frac{1}{(\sigma_v^2)^{\frac{m_1}{2}}} \exp\left(-\frac{v^{(1)T}v^{(1)}}{2\sigma_v^2}\right), \qquad (8.6)
\end{aligned}$$

where $R_1 = I - M_1(M_1^TM_1)^{-1}M_1^T$.

Before we proceed, let us state the following lemma.

Lemma 8.5 *The following results hold: (a)* $\mathrm{rank}\,[R_1Z_1^{(1)}] = \mathrm{rank}\begin{pmatrix} M_1 & Z_1^{(1)}\end{pmatrix} - \mathrm{rank}(M_1)$, *(b)* $\mathrm{rank}(R_2) = n_1^* - \mathrm{rank}\begin{pmatrix} M_1 & Z_1^{(1)}\end{pmatrix}$, *where* $R_2 = I - Proj\begin{pmatrix} M_1 & Z_1^{(1)}\end{pmatrix}$.

Proof of Lemma 8.5 is discussed in Section 8.3. We have, $\mathrm{rank}\begin{pmatrix} M_1 & Z_1^{(1)}\end{pmatrix} = \mathrm{rank}(M_1) + \mathrm{rank}(Z_1^{(1)}) - 1 = p_1 + m_1 - 1$. Hence, we have $\mathrm{rank}(R_2) = n_1^* - \mathrm{rank}\begin{pmatrix} M_1 & Z_1^{(1)}\end{pmatrix} = n_1^* - (p_1 + m_1 - 1) = (n_1^* - p_1 - m_1) + 1 \geq 1$ (by (a) of Lemma 8.4). Thus R_2 is positive-semidefinite and $y_1^TR_2y_1 > 0$ with probability 1.

Let $Q_1 = Z_1^{(1)T}R_1Z_1^{(1)}$. Since R_1 is symmetric and idempotent, $\mathrm{rank}(Q_1) = \mathrm{rank}[R_1Z_1^{(1)}] = \mathrm{rank}\begin{pmatrix} M_1 & Z_1^{(1)}\end{pmatrix} - \mathrm{rank}(M_1) = m_1 + p_1 - 1 - p_1 = m_1 - 1 = t_1$ (say). Let P_1 be

an orthogonal matrix such that $P_1^T Q_1 P_1 = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{t_1}, 0, \ldots, 0)$, where, $\lambda_1 \geqslant \lambda_2 \cdots \geqslant \lambda_{t_1} > 0$ are the positive eigenvalues of Q_1.

Let $\hat{v}^{(1)}$ denote a minimizer of $(y_1 - Z_1^{(1)} v^{(1)})^T R_1 (y_1 - Z_1^{(1)} v^{(1)})$ wrt $v^{(1)}$. We use the transformation $w{=}P_1 v^{(1)}$ in (8.6).

$$\begin{aligned}\int \tilde{\varphi}(y_1, y_{211}, y_2^*, v, \rho_1, \rho_2, \sigma_1^2, \sigma_2^2, \sigma_v^2, p_e)\, \mathrm{d}y_2^*\, \mathrm{d}\rho_2\, \mathrm{d}\sigma_2^2\, \mathrm{d}v^{(2)}\, \mathrm{d}\rho_1 \\ = C \times \frac{1}{(\sigma_1^2)^{\frac{n_1^* - p_1^* + 2}{2}}} \exp\left\{-\frac{w^T w}{2\sigma_v^2}\right\} \times \frac{1}{(\sigma_v^2)^{\frac{m_1}{2}}} \exp\left\{-\frac{y_1^T R_2 y_1}{2\sigma_1^2}\right\} \\ \times \exp\left\{-\frac{\sum_{j=1}^{t_1} \lambda_j (w_j - \hat{w}_j)^2}{2\sigma_1^2}\right\}, \end{aligned} \tag{8.7}$$

where $P_1 \hat{v}^{(1)} = \hat{w}$. We integrate out $w_{t_1+1}, \ldots, w_{m_1}$:

$$\begin{aligned}&\int \tilde{\varphi}(y_1, y_{211}, y_2^*, v, \rho_1, \rho_2, \sigma_1^2, \sigma_2^2, \sigma_v^2, p_e)\, \mathrm{d}y_2^*\, \mathrm{d}\rho_2\, \mathrm{d}\sigma_2^2\, \mathrm{d}v^{(2)}\, \mathrm{d}\rho_1 \prod_{k=t_1+1}^{m_1} \mathrm{d}w_k \\ &= C \times \frac{1}{(\sigma_1^2)^{\frac{n_1^* - p_1^* + 2}{2}}} \exp\left\{-\frac{y_1^T R_2 y_1}{2\sigma_1^2}\right\} \exp\left\{-\frac{\sum_{j=1}^{t_1} \lambda_j (w_j - \hat{w}_j)^2}{2\sigma_1^2}\right\} \frac{1}{(\sigma_v^2)^{\frac{t_1}{2}}} \exp\left\{-\frac{\sum_{j=1}^{t_1} w_j^2}{2\sigma_v^2}\right\}.\end{aligned}$$

We integrate the last equation with respect to σ_1^2 and σ_v^2 using inverse gamma density integration result. By the conditions $n_1^* > p_1 + m_1$ and $m_1 > 3$, the shape parameters will be positive. Hence after substantial simplifications,

$$\begin{aligned}\int \tilde{\varphi}(y_1, y_{211}, y_2^*, v, \rho_1, \rho_2, \sigma_1^2, \sigma_2^2, \sigma_v^2, p_e)\, \mathrm{d}y_2^*\, \mathrm{d}\rho_2\, \mathrm{d}\sigma_2^2\, \mathrm{d}v^{(2)}\, \mathrm{d}\rho_1 \prod_{k=t_1+1}^{m} \mathrm{d}w_k\, \mathrm{d}\sigma_1^2\, \mathrm{d}\sigma_v^2 \\ \leq C \times \frac{1}{\left\{y_1^T R_2 y_1 + \lambda_{t_1} \sum_{j=1}^{t_1} (w_j - \hat{w}_j)^2\right\}^{\frac{n_1^* - p_1}{2}}} \times \frac{1}{\left(\sum_{j=1}^{t_1} w_j^2\right)^{\frac{t_1 - 2}{2}}}. \end{aligned} \tag{8.8}$$

Let us denote $\sum_{j=1}^{t_1} \hat{w_j}^2$ by d^2, then for any $\epsilon > 0$, and positive constants C_1 and C_2,

$$\int \tilde{\varphi}(y_1, y_{211}, y_2^*, v, \rho_1, \rho_2, \sigma_1^2, \sigma_2^2, \sigma_v^2, p_e)\, \mathrm{d}y_2^*\, \mathrm{d}\rho_2\, \mathrm{d}\sigma_2^2\, \mathrm{d}v^{(2)}\, \mathrm{d}\rho_1 \prod_{k=t_1+1}^{m_1} \mathrm{d}w_k\, \mathrm{d}\sigma_1^2\, \mathrm{d}\sigma_v^2$$

$$\leq C_1 \times \frac{1}{\{y_1^T R_2 y_1\}^{\frac{n_1^*-p_1}{2}}} \times \frac{\mathrm{I}\Big(\sum_{j=1}^{t_1} w_j^2 \leq 2d^2 + \epsilon\Big)}{\Big(\sum_{j=1}^{t_1} w_j^2\Big)^{\frac{t_1-2}{2}}}$$

$$+ C_2 \times \frac{1}{\lambda_{t_1}^{\frac{(n_1^*-p_1)}{2}} \Big[\frac{1}{2}\sum_{j=1}^{t_1} w_j^2 - d^2\Big]^{\frac{n_1^*-p_1}{2}}} \times \frac{\mathrm{I}\Big(\sum_{j=1}^{t_1} w_j^2 > 2d^2 + \epsilon\Big)}{\Big(\sum_{j=1}^{t_1} w_j^2\Big)^{\frac{t_1-2}{2}}}. \quad (8.9)$$

Using the polar transformation for $w_1, \ldots, w_{t_1}$, it follows after substantial simplifications, the integrability of the rhs of (8.9) follows.

So far we have proved that any arbitrary typical term in (8.1) satisfying conditions (a), (b) and (c) is integrable. Hence, we can conclude, $f(y, v, \beta, \sigma_1^2, \sigma_2^2, \sigma_v^2, p_e)$ (in (8.1)) is integrable with respect to $v, \beta, \sigma_1^2, \sigma_2^2, \sigma_v^2, p_e$ if condition (a), (b) and (c) are satisfied. □

8.3 Proof of the Lemmas

Proof of Lemma 8.4:

Proof At first we note that at least one of these two conditions $n_1^* \geq m_1 + p_1$ and $n_2^* \geq m_2 + p_2$ holds. In order to establish that, let us assume, $n_1^* \leq m_1 + p_1 - 1$ and $n_2^* \leq m_2 + p_2 - 1$, i.e., $n = n_1^* + n_2^* \leq m_1 + p_1 + m_2 + p_2 - 2 < 2m + 2p - 1$, which contradicts to our assumption that $n = 2m + 2p - 1$.

Note that, $n_1^* = \sum_{i \in S_1} n_{i1} \leq 2m_1 + 2p - 1$. If possible, let $n_1^* > 2m_1 + 2p - 1$, that is, m_1 small areas have more than $2m_1 + 2p - 1$ observations. Since we previously assumed that (in Theorem 8.1) $n_i \geq 2$, for all i. Hence the remaining $(m - m_1)$ small areas have at least $2(m - m_1)$ observations overall. Therefore, $n = n_1^* + n_2^* > 2m_1 + 2p - 1 + 2(m - m_1) = 2m + 2p - 1$, which is a contradiction to the previous assumption that $n = 2m + 2p - 1$. With similar arguments we can establish $n_2^* \leq 2m_2 + 2p - 1$.

Now we consider various scenarios and prove that either (a) or (b) holds.

Case - I: $n_1^* \geq m_1 + p_1$ and $m_1 \leq 3$.

We know $n_1^* \leq 2m_1 + 2p - 1$. Hence, $n_1^* \leq 6 + 2p - 1 = 2p + 5$. Also, $n_2^* = n - n_1^* \geq n - (2p + 5) = 2m - 6 \geq m + p \geq m_2 + p_2$.

Now, let us assume, $m_2 \leq 3$, $n_2^* \leq 2m_2 + 2p - 1 \Rightarrow n_2^* \leq 2p + 6 \leq p - 1 + m < m + p$, which contradicts to our earlier assertion $n_2^* \geq m + p$. Hence $m_2 > 3$. Therefore, $n_2^* \geq m_2 + p_2$, $m_2 > 3$; i.e. condition (b) holds.

Case - II: $n_2^* \geq m_2 + p_2$ and $m_2 \leq 3$. With the similar arguments, as in Case - I, it can be shown that condition (a) holds in this case.

Case - III: $n_2^* < m_2 + p_2$, $m_2 > 3$ or $m_2 \leq 3$. In this case, $n_1^* = n - n_2^* > 2m + 2p - 1 - (m_2 + p_2) \geq 2m + 2p - m - p - 1 \geq m_1 + p_1 - 1$. Hence, $n_2^* < m_2 + p_2 \Longrightarrow n_1^* \geq m_1 + p_1$. Again, let us assume, $m_1 \leq 3$. Now, $n_1^* \leq 2m_1 + 2p - 1 \leq 2p + 5$ $\Longrightarrow n = n_1^* + n_2^* \leq (2p + 5) + (m_2 + p_2 - 1) \leq 2p + 5 + m + p - 1 = 3p + m + 4 \Rightarrow n = 2m + 2p - 1 \leq 3p + m + 4 \Longleftrightarrow m \leq p + 5$, which contradicts to our earlier assumption that $m \geq p + 6$. Therefore $m_1 \not\leq 3$ in this case, i. e. $m_1 > 3$. Hence, $n_2^* < m_2 + p_2 \Rightarrow n_1^* \geq m_1 + p_1$ and $m_1 > 3$, i.e., condition (a) holds.

Case - IV: $n_1^* < m_1 + p_1$, $m_1 > 3$ or $m_1 \leq 3$. It can be proved that condition (2) will hold in this scenario. Hence, under the proposed model at least one of the conditions stated will hold. □

Proof of Lemma 8.5

Proof (a) Here, $R_1 = I - M_1(M_1^T M_1)^{-1} M_1^T \Rightarrow R_1 Z^{(1)} = Z_1^{(1)} - M_1(M_1^T M_1)^{-1} M_1^T Z_1^{(1)} \Rightarrow M_1^T(R_1 Z_1^{(1)}) = 0 \Rightarrow$ columns of M_1 are orthogonal to the columns of $R_1 Z_1^{(1)}$. Therefore,

$$\text{rank}\Big(R_1 Z_1^{(1)} \quad M_1\Big) = \text{rank}(R_1 Z_1^{(1)}) + \text{rank}(M_1).$$

Now,

$$\Big(R_1 Z_1^{(1)} \quad M_1\Big) = \Big(Z_1^{(1)} \quad M_1\Big) \begin{pmatrix} I & 0 \\ -(M_1^T M_1)^{-1} M_1^T Z_1^{(1)} & I \end{pmatrix}$$

$\Rightarrow \text{rank}\Big(R_1 Z_1^{(1)} \quad M_1\Big) = \text{rank}\Big(Z_1^{(1)} \quad M_1\Big)$ (since $\begin{pmatrix} I & 0 \\ -(M_1^T M_1)^{-1} M_1^T Z_1^{(1)} & I \end{pmatrix}$ is non singular) $\Rightarrow \text{rank}(R_1 Z_1^{(1)}) = \text{rank}\Big(Z_1^{(1)} \quad M_1\Big) - \text{rank}(M_1) = \text{rank}\Big(M_1 \quad Z_1^{(1)}\Big) - \text{rank}(M_1)$.

(b) $R_2 = R_1 - R_1 Z_1^{(1)} (Z_1^{(1)T} R_1 Z_1^{(1)})^- Z_1^{(1)T} R_1$, R_2 is idempotent.
Therefore, $\text{rank}(R_2) = \text{rank}(R_1) - \text{rank}(R_1 Z_1^{(1)}) = (n_1^* - \text{rank}(M_1)) - \text{rank}\begin{pmatrix} M_1 & Z_1^{(1)} \end{pmatrix}$
$- \text{rank}(M_1) = n_1^* - \text{rank}\begin{pmatrix} Z_1^{(1)} & M_1 \end{pmatrix} = n_1^* - \text{rank}\begin{pmatrix} M_1 & Z_1^{(1)} \end{pmatrix}$. □

References

Hobert, J. & Casella, G. (1996). Effect of improper priors on Gibbs sampling in hierarchical linear mixed models, *Journal of the American Statistical Association*, **91**, 1461–1473.