A-ComVar: A Flexible Extension of Common Variance Designs

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Abstract

We consider nonregular fractions of factorial experiments for a class of linear models. These models have a common general mean and main effects, however they may have different 2-factor interactions. Here we assume for simplicity that 3-factor and higher order interactions are negligible. In the absence of *a priori* knowledge about which interactions are important, it is reasonable to prefer a design that results in equal variance for the estimates of all interaction effects to aid in model discrimination. Such designs are called common variance designs and can be quite challenging to identify without performing an exhaustive search of possible designs. In this work, we introduce an extension of common variance designs called approximate common variance, or A-ComVar designs. We develop a numerical approach to finding A-ComVar designs that is much more efficient than an exhaustive search. We present the types of A-ComVar designs that can be found for different number of factors, runs, and interactions. We further demonstrate the competitive performance of both common variance and A-ComVar designs with Plackett-Burman designs for model selection using simulation.

Keywords: Class of Models, Model Identification, Common Variance, Placket-Burman, Adaptive Lasso, Approximate Common Variance, Genetic Algorithm

1 Introduction

Fractional factorial designs are widely used in many scientific investigations because they provide a systematic and statistically valid strategy for studying how multiple factors impact a response variable through main effects and interactions. When several factors are to be tested, often the experimenter does not know which factors have important interactions. Instead, the experimenter will need to perform model selection after conducting the experiment to identify important interactions. Generally this process will involve fitting different models under consideration and examining statistical significance of the interaction terms. Some techniques have been developed concerning finding efficient fractional factorial plans for this purpose. There is a rich literature on identification and discrimination to find the model best describing the data (Srivastava, 1976; Srivastava and Ghosh, 1976; Srivastava and Gupta, 1979).

Borrowing notation from Ghosh and Chowdhury (2017), consider the following class of s candidate models for describing the relationship between p factors and the $n \times 1$ vector of observations $\boldsymbol{y}$,

$$\begin{aligned} E(\boldsymbol{y}) &= \beta_0 \boldsymbol{j}_n + \boldsymbol{X}_1 \boldsymbol{\beta}_1 + \boldsymbol{X}_2^{(i)} \boldsymbol{\beta}_{2i},\ i = 1, \ldots, s \qquad (1) \\ Var(\boldsymbol{y}) &= \sigma^2 \boldsymbol{I}, \end{aligned}$$

where n is the number of runs, β_0 is the general mean, $\boldsymbol{j}_n$ is a vector of ones, $\boldsymbol{\beta}_1$ is the vector of p main effects that are common in all s models. The other parameters, $\boldsymbol{\beta}_{2i}$, are specific for the ith model and hence $\boldsymbol{\beta}_{2i} \neq \boldsymbol{\beta}_{2i'}$ for $i \neq i'$, $i = 1, \ldots, s$. We call these parameters "uncommon parameters." The design matrices $\boldsymbol{X}_1$ and $\boldsymbol{X}_2^{(i)}$ correspond to the main effects and i^{th} set of two factor interactions, respectively.

Under the above setup, model selection consists of identifying the correct i from the s candidate models. This process is complicated by the fact that the variance estimates for the uncommon parameters are generally not the same, which can pre-bias the experiment towards identifying certain interactions as significant over others, i.e. making some i more likely to be selected than others regardless of the true underlying model. To address this issue, Ghosh and Flores (2013) introduced the notion of common variance designs for a single uncommon parameter. These designs estimate the uncommon parameter in all models with equal variance, which is desirable in the absence of any *a priori* information about the true model. Ghosh and Chowdhury (2017) generalized this concept of common variance to k ($k \geq 1$) uncommon parameters in each model in the class. Under the situation of $k > 1$, Ghosh and Chowdhury (2017) defined a common variance design to be the one satisfying $|\boldsymbol{X}^{(i)\prime}\boldsymbol{X}^{(i)}|$ to be a constant, for all i, $\boldsymbol{X}^{(i)} = \left(\boldsymbol{j}_n, \boldsymbol{X}_1, \boldsymbol{X}_2^{(i)}\right)$.

The concept of variance-balancedness is not totally new. Different types of "variance-balanced designs" estimating all or some of the treatment-contrasts with identical variance were developed by Calvin (1986), Cheng (1986), Gupta and Jones (1983), Hedayat and Stufken (1989), Khatri (1982), Mukerjee and Kageyama (1985), among others.

While common variance designs have been identified for 2 and 3 level factorial experiments with a single 2-factor interaction (Ghosh and Flores, 2013; Ghosh and Chowdhury, 2017), it remains to develop a method which can find them for general number of factors and interactions. To date, these designs have been found using exhaustive searches, which becomes prohibitively expensive as the number of factors and runs increases. This leads us to introduce approximate common variance (A-ComVar) designs, which relax the requirement that the variance of the uncommon parameters be exactly equal. We introduce an objective function that allows us to rank designs under consideration, and we develop a genetic algorithm for searching for these designs. Moreover, we investigate the performance of both common variance and A-ComVar designs for model selection using the adaptive lasso regression technique in simulation (Kane and Mandal, 2019). We find comparable performance of common variance and A-ComVar designs to Placket-Burman designs, which further demonstrates the usefulness of designs that prioritize having a similar variance for the uncommon parameters in the model.

The rest of the article is organized as follows. In Section 2 we present the current state of knowledge for both two level and three level common variance designs. For three level designs we also present the exhaustive search result for $m = 3$. In Section 3 we illustrate the advantage of common variance designs for model selection via a comparison of the performance of common variance designs with some popular non-regular designs such as Plackett-Burman designs using simulations and the adaptive lasso regression technique. In Section 4 we introduce our numerical approach for finding A-ComVar designs. In Section 5 we conduct numerical simulations examining the algorithm's ability to find common variance designs as we increase the number of factors and number of interactions in the model. We also conduct simulations comparing the model selection capabilities of A-ComVar designs to Placket-Burman designs. We conclude the article with some discussion in Section 6.

2 Common Variance Designs

2.1 2 Level Designs

The term "common variance" for the class of variance balanced designs was first introduced in Ghosh and Flores (2013). As a more stringent criteria, the authors also introduced the concept of optimum common variance (OPTCV), which is satisfied by designs having the smallest value of common variance in a class of common variance designs with m factors and n runs. Several characterizations of common variance and optimal common variance designs were presented that provide efficient ways for checking the common variance or OPTCV property of a given design. These characterizations were obtained in terms of the projection matrix, eigenvalues of the model matrix, balancedness, and orthogonal properties of the designs. In Corollary 1 of Ghosh and Flores (2013), they stated one sufficient condition of common variance designs in terms of equality of the vectors of eigen values of $\boldsymbol{X}^{(i)\prime}\boldsymbol{X}^{(i)}$, $\boldsymbol{X}^{(i)} = \left(\boldsymbol{j}_n, \boldsymbol{X}_1, \boldsymbol{X}_2^{(i)}\right)$, for all i. We present one design in Table 2 from Ghosh and Flores (2013) for $m = 5$ and $n = 12$, that has identical vectors of eigen values for all i. In Section 3 we compare the performance of this particular design with that of Plackett-Burman designs for model selection to demonstrate further usefulness of such designs.

In their work, Ghosh and Flores (2013) presented several general series of designs with the common variance property. For example, they identified two fold-over designs with the common variance property with m factors and $n = 2m$ and $n = 2m + 2$ runs respectively:

$$d_m^{(2m)} = \begin{bmatrix} 2\boldsymbol{I}_m - \boldsymbol{J}_m \\ -2\boldsymbol{I}_m + \boldsymbol{J}_m \end{bmatrix}.$$

$$d_m^{(2m+2)} = \begin{bmatrix} \boldsymbol{j}'_m \\ -\boldsymbol{j}'_m \\ 2\boldsymbol{I}_m - \boldsymbol{J}_m \\ -2\boldsymbol{I}_m + \boldsymbol{J}_m \end{bmatrix}.$$

As reported in Ghosh and Flores (2013), both of these designs are balanced arrays of full strength and orthogonal arrays of strength 1, $\forall\ m$. Moreover, the design $\boldsymbol{d}_m^{(2m)}$ is OPTCV for $m = 4$ and $\boldsymbol{d}_m^{(2m+2)}$ is OPTCV for $m = 3$.

2.2 3 Level Designs

Ghosh and Chowdhury (2017) presented common variance designs for 3^m fractional factorial experiments. Consider the following model for a 3^m factorial experiment, with one two-factor interaction effect in the model, i.e. $k = 1$:

$$E(\boldsymbol{y}) = \beta_0 \boldsymbol{j}_n + \boldsymbol{X}_1 \boldsymbol{\beta}_1 + \boldsymbol{X}_2^{(i)} \beta_{2i},\ Var(\boldsymbol{y}) = \sigma^2.$$

A design for such an experiment would have the common variance property iff $\frac{Var(\hat{\beta}_{2i})}{\sigma^2}$ is constant for all $i = 1, \ldots, \binom{4m}{2}$.

Ghosh and Chowdhury (2017) presented two general series of 3^m fractional factorial common variance designs d_1 and d_2 with n runs. The design d_1 has a common variance value given by $\frac{Var\left(\hat{\beta}_2^{(i)}\right)}{\sigma^2} = \frac{2-m+m^2}{9}$, for $m \geq 2$ and $n = 2m + 2$ runs, while design d_2 has a common variance value given by $\frac{Var\left(\hat{\beta}_2^{(i)}\right)}{\sigma^2} = \frac{m}{9(m-2)}$, for $m \geq 3$ and $n = 3m$. Also, the design d_1 is efficient common variance (ECV, as termed in Ghosh and Chowdhury (2017)) design for $m = 2$, and design d_2 is ECV for $m = 3$.

Ghosh and Chowdhury (2017) also presented several sufficient conditions for general fractional factorial designs to have the common variance property, including the special case for 3^m designs in terms of the projection matrix of the design and the columns of two-factor interaction. For example, a design is common variance if (i) $\boldsymbol{P}\boldsymbol{X}_2^{(i_1)} = \boldsymbol{P}\boldsymbol{X}_2^{(i_2)}$, for $i_1, i_2 \in \{1, \ldots, s\}$, where $\boldsymbol{P}$ is the projection matrix defined as $\boldsymbol{I}_n - \boldsymbol{X}_1 \left(\boldsymbol{X}'_1 \boldsymbol{X}_1\right)^{-1} \boldsymbol{X}'_1$, and $\boldsymbol{X}_1$ contains the columns corresponding to the general mean and main effects from the model matrix $\boldsymbol{X}^{(i)} = \left(\boldsymbol{j}_n, \boldsymbol{X}_1, \boldsymbol{X}_2^{(i)}\right)$, and $\boldsymbol{X}_2^{(i)}$

corresponds to the i^{th} two-factor interaction, $i = 1, \ldots, \binom{4m}{2}$. Another set of sufficient conditions for having common variance is, for, $i_1, i_2 \in \{1, \ldots, s\}$, (i) $\left(\boldsymbol{X}_2^{(i_1)} \pm \boldsymbol{X}_2^{(i_2)}\right)$ belongs to the column space of $\boldsymbol{X}_1$ and (ii) $\boldsymbol{X}_2^{(i_2)} = \boldsymbol{F}\boldsymbol{X}_2^{(i_1)}$ holds, where the permutation matrix $\boldsymbol{F}$ obtained from the identity matrix satisfies $\boldsymbol{F}'\boldsymbol{P}\boldsymbol{F} = \boldsymbol{P}$.

For 3^3 fractional factorial experiment, Chowdhury (2016) conducted a complete search of common variance designs for $n = 8$ to $n = 27$, since $n = 8$ is the minimum number of runs needed to estimate all the parameters (one general mean + 6 main effects + one 2-factor interaction effect). The results of this search are presented in Table 1. The complete search revealed that common variance designs only exist for $n = 8, 9, 10, 11$ for 3^3 factorial experiments. For each of the runs multiple groups of common variance designs were obtained, having different common variance values, among which 32 designs for $n = 11$; 48 designs for $n = 10$; 8256 designs for $n = 9$; and 9600 designs for $n = 8$, are the efficient common variance designs giving the minimum value of common variance in the respective classes.

Table 1: Complete search results for finding common variance designs for 3^3 factorial experiments.

n	Possible Designs $=\binom{27}{n}$	Satisfying Rank Condition	No. of Non-CV designs	No. of CV designs	Groups No. of CV designs in each group	CV value
11	13,037,895	6,926,898	6,924,772	2,096	32	0.2151
					2,064	0.2222
10	8,436,285	2,792,387	2,775,747	16,640	48	0.2564
					48	0.2667
					16	0.2837
					16,512	0.2963
					16	0.4
9	4,686,825	636,348	588,348	48,000	8,256	0.3333
					32	0.381
					13,056	0.4167
					26,640	0.4444
					16	0.5
8	2,220,075	49,628	23,340	26,288	9,600	0.6667
					16,688	0.8889

3 Common Variance Designs for Model Selection - An Example

In this section we use simulation studies to demonstrate the advantages of having common variance designs by comparing the performance of the design $\boldsymbol{d}_5^{(12)}$ in Table 2 with two sets of five factors from a Plackett-Burman design (Plackett and Burman, 1946) given by factors $A - E$ and $F - J$ in Table 3.

Table 2: Design $\boldsymbol{d}_5^{(12)}$ with common variance for 5 factors and 12 runs used for comparison with the Plackett-Burman design.

A	B	C	D	E
-1	-1	-1	-1	1
1	1	-1	-1	-1
1	-1	1	-1	-1
1	-1	-1	1	-1
-1	1	1	-1	-1
-1	1	-1	1	-1
-1	-1	1	1	-1
1	1	1	1	-1
1	1	1	-1	1
1	1	-1	1	1
1	-1	1	1	1
-1	1	1	1	1

Table 3: Plackett-Burman Design with 11 factors and 12 runs

A	B	C	D	E	F	G	H	I	J	K
1	1	-1	1	1	1	-1	-1	-1	1	-1
1	-1	1	1	1	-1	-1	-1	1	-1	1
-1	1	1	1	-1	-1	-1	1	-1	1	1
1	1	1	-1	-1	-1	1	-1	1	1	-1
1	1	-1	-1	-1	1	-1	1	1	-1	1
1	-1	-1	-1	1	-1	1	1	-1	1	1
-1	-1	-1	1	-1	1	1	-1	1	1	1
-1	-1	1	-1	1	1	-1	1	1	1	-1
-1	1	-1	1	1	-1	1	1	1	-1	-1
1	-1	1	1	-1	1	1	1	-1	-1	-1
-1	1	1	-1	1	1	1	-1	-1	-1	1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

Our simulation sought to measure how well each design could identify the true underlying model, which corresponds to identifying the significant active effects in the model without also including any inert effects. To do this, we considered both "big" (B) and "small" (S) effects for a series of different models of varying complexity. For the "big" effect we randomly generated the coefficient from $U(0.5, 1.5)$, and for the "small" effect the coefficients are randomly generated from $U(0.1, 0.3)$. For each design under consideration, we generated the response values from the specified model. For example, to fix ideas consider the model specified by the 9th row of Table 4. For this simulation, data were generated from $y = \beta_0 + \beta_1 F_1 + \beta_2 F_2 + \beta_{12} F_1 F_2 + \epsilon$, where F_1 and F_2 are the factors A and B from Table 2 for the common variance design and Table 3 for the

Plackett-Burman design. We used the adaptive lasso (Kane and Mandal, 2019) to fit the model. The model was then said to be correctly identified if the only significant effects were for F_1, F_2, and their interaction. We chose the adaptive lasso method of (Kane and Mandal, 2019) because they showed that this technique is suitable for identifying the correct model for designs with complex aliasing and that it outperforms other popular variable selection methods including the Dantzig Selector (Candes et al., 2007), LARS (Yuan et al., 2007), and the Nonnegative Garotte estimator (Breiman, 1995; Yuan et al., 2009). We repeated this process 100 times for each model setup and reported the percentage of times the true underlying model was correctly identified.

Tables 4 and 5 display the percentage of times the correct model is identified by the common variance design $\boldsymbol{d}_5^{(12)}$ and the Placket-Burman design presented in Table 3, under different factorial effect size combinations and varied choices of the variance (σ^2) of the noise. In Table 4 we compare the performance of model selection between $\boldsymbol{d}_5^{(12)}$ and the design with factors $A - E$ from the Plackett-Burman design, while in Table 5 we compare between $\boldsymbol{d}_5^{(12)}$ and the design with factors $F - J$ from the same design given in Table 3.

As we observe from Tables 4−5, the model selection performance of the two designs are fairly comparable across different models and sizes. However, we do observe that as the model variance σ^2 increases the percentage of correct model selection by $\boldsymbol{d}_5^{(12)}$ is much higher than that of the Plackett-Burman designs, and this difference is quite striking for larger variances.

4 Identifying Common Variance Designs

4.1 Challenges in Numerically identifying Common Variance Designs

Ghosh and Flores (2013) and Ghosh and Chowdhury (2017) presented some general series of designs satisfying the common variance property for two and three level factorial experiments obtained via exhaustive searches of the design space. Such searches become extremely computationally challenging as the number of factors increases. For example, for a 3^3 factorial experiment with one 2-factor interaction ($k = 1$) the possible set of candidate designs with 8 runs is $\binom{27}{8} = 2220075$, with 9 runs is $\binom{27}{9} = 4686825$, with 10 runs is $\binom{27}{10} = 8436285$, and so on. For a 3^4 factorial experiment, the cardinality of this set increases to $\binom{81}{10} = 1.878392 \times 10^{12}$, even for the designs with the smallest possible number of runs. This rapid growth in the size of the search space makes exhaustive searches

Table 4: Percentage of Correctly Identified Models for design $\boldsymbol{d}_5^{(12)}$ and Plackett-Burman design (PB) with factors $A - E$.

	Model	Size	0.1		0.25		0.5		0.75		1	
			CV	PB	CV	PB	CV	PB	CV	PB	CV	PB
1	F_1	B	100	100	0	1	22	34	14	5	6	7
2	F_1	S	92	86	0	1	16	6	99	100	5	0
3	$F_1 + F_2$	$B + B$	60	68	100	100	3	5	26	27	1	0
4	$F_1 + F_2$	$B + S$	56	60	92	81	2	2	9	3	100	62
5	$F_1 + F_2$	$S + S$	56	49	53	55	65	76	5	8	55	3
6	$F_1 + F_1F_2$	$B + B$	82	88	30	26	30	10	6	0	9	1
7	$F_1 + F_1F_2$	$B + S$	9	24	49	27	11	6	25	45	4	0
8	$F_1 + F_1F_2$	$S + S$	6	15	100	97	17	16	12	15	1	1
9	$F_1 + F_2 + F_1F_2$	$B + B + B$	3	10	26	15	7	4	3	1	94	4
10	$F_1 + F_2 + F_1F_2$	$B + B + S$	3	3	6	13	89	96	0	0	9	2
11	$F_1 + F_2 + F_1F_2$	$B + S + B$	100	100	6	2	17	9	0	0	5	0
12	$F_1 + F_2 + F_1F_2$	$B + S + S$	70	68	4	1	2	6	91	55	3	0
13	$F_1 + F_2 + F_1F_2$	$S + S + S$	49	44	92	88	2	0	7	9	0	0
14	$F_1 + F_2 + F_3$	$B + B + B$	20	22	48	25	0	0	0	1	85	12
15	$F_1 + F_2 + F_3$	$B + B + S$	19	16	11	8	89	84	0	0	10	4
16	$F_1 + F_2 + F_3$	$B + S + S$	100	39	5	2	15	7	0	0	5	0
17	$F_1 + F_2 + F_3$	$S + S + S$	14	10	1	0	2	2	100	94	0	0
18	$F_1 + F_2 + F_1F_3$	$B + B + B$	11	11	100	100	0	1	79	68	0	1
19	$F_1 + F_2 + F_1F_3$	$B + B + S$	8	1	85	87	0	1	38	5	87	33
20	$F_1 + F_2 + F_1F_3$	$B + S + B$	2	2	57	36	100	100	13	0	7	1
21	$F_1 + F_2 + F_1F_3$	$B + S + S$	86	79	24	25	74	68	5	1	0	0
22	$F_1 + F_2 + F_1F_3$	$S + B + S$	14	20	28	1	31	18	46	78	0	0
23	$F_1 + F_2 + F_1F_3$	$S + S + S$	0	1	100	65	23	16	41	2	0	0

for common variance designs impossible for anything but small design problems.

In light of the difficulty in finding common variance designs, we introduce a class of approximate common variance (A-ComVar) designs. Instead of having exactly equal variance for the uncommon parameters for the s models under consideration, A-ComVar designs try to minimize the ratio of the minimum variance to the maximum variance. In doing so, they contain common variance designs as a sub-case where the minimum variance is exactly equal to the maximum variance. In relaxing the requirement that the variances be exactly equal, we are able to develop an objective function and algorithm for identifying these A-ComVar designs without performing an exhaustive search.

Table 5: Percentage of Correctly Identified Models for design $\boldsymbol{d}_5^{(12)}$ and Plackett-Burman design (PB) with factors $F - J$.

	Model	Size	0.1		0.25		0.5		0.75		1	
			CV	PB	CV	PB	CV	PB	CV	PB	CV	PB
1	F_1	B	100	100	1	1	55	55	7	3	7	2
2	F_1	S	87	80	2	1	15	7	41	99	1	1
3	$F_1 + F_2$	$B + B$	73	62	100	100	8	2	61	34	1	0
4	$F_1 + F_2$	$B + S$	50	40	83	96	1	0	4	6	84	82
5	$F_1 + F_2$	$S + S$	40	42	62	62	57	100	2	4	51	8
6	$F_1 + F_1F_2$	$B + B$	80	60	50	47	65	46	3	2	8	1
7	$F_1 + F_1F_2$	$B + S$	39	46	43	14	13	5	99	100	5	0
8	$F_1 + F_1F_2$	$S + S$	9	7	96	95	2	3	17	23	4	1
9	$F_1 + F_2 + F_1F_2$	$B + B + B$	11	8	59	48	4	7	2	1	4	56
10	$F_1 + F_2 + F_1F_2$	$B + B + S$	9	3	10	15	99	95	0	2	21	0
11	$F_1 + F_2 + F_1F_2$	$B + S + B$	100	100	4	7	24	14	0	0	1	0
12	$F_1 + F_2 + F_1F_2$	$B + S + S$	86	73	3	4	2	6	87	60	5	0
13	$F_1 + F_2 + F_1F_2$	$S + S + S$	43	45	78	72	0	0	2	2	0	0
14	$F_1 + F_2 + F_3$	$B + B + B$	37	8	36	18	1	1	0	0	75	68
15	$F_1 + F_2 + F_3$	$B + B + S$	32	28	13	7	87	75	0	0	17	12
16	$F_1 + F_2 + F_3$	$B + S + S$	96	100	5	4	17	7	0	0	1	1
17	$F_1 + F_2 + F_3$	$S + S + S$	47	41	4	3	5	2	100	84	0	0
18	$F_1 + F_2 + F_1F_3$	$B + B + B$	11	5	100	100	0	0	83	5	0	2
19	$F_1 + F_2 + F_1F_3$	$B + B + S$	1	5	88	81	0	0	26	16	78	12
20	$F_1 + F_2 + F_1F_3$	$B + S + B$	1	4	47	51	100	100	20	2	12	0
21	$F_1 + F_2 + F_1F_3$	$B + S + S$	79	77	24	26	80	67	13	0	0	0
22	$F_1 + F_2 + F_1F_3$	$S + B + S$	6	13	23	8	35	40	100	96	0	0
23	$F_1 + F_2 + F_1F_3$	$S + S + S$	2	1	95	99	13	2	69	13	0	0

4.2 Proposed Algorithm: Genetic Algorithm for Finding A-ComVar Designs

In this section we propose to use a genetic algorithm to identify A-ComVar Designs. We start by defining an objective function that seeks to quantify our goal.

$$f(d;\phi) = \frac{1/\min_i\{var(\hat{\beta}_{2i})\}}{1 + \phi \times \left[\max_i\{var(\hat{\beta}_{2i})\} - \min_i\{var(\hat{\beta}_{2i})\}\right]}, \tag{2}$$

where β_{2i} corresponds to the interaction effect for the ith model. The value of the objective function increases as the variance of the estimates decreases through the numerator, encouraging designs with small variances for the interaction terms. However, this value is also strongly penalized towards zero as the minimum and maximum variances move apart. The strength of this penalty is controlled by the tuning parameter ϕ, which we recommend setting to a large value. In our experiments we found

$\phi = 1000$ to be adequate. Thus taken together the numerator allows us to differentiate between designs with common variance to select the better one, and the denominator encourages common variance designs by penalizing differing variance under alternative models under consideration.

This maximization approach will prefer A-ComVar designs with exactly common variance. Of course, in many experimental situations a common variance design may not exist. For example in the exhaustive search, Chowdhury (2016) found that common variance designs did not exist for 3^3 experiments for 13 runs. This leads us to the principal advantage of our approach: when a common variance design does not exist we can still find designs with variance that is as close as possible to being equal. To assess the quality of an A-ComVar design, we define the A-ComVar ratio

$$r_{ACV} = \frac{\min_i \{var(\hat{\beta}_{2i})\}}{\max_i \{var(\hat{\beta}_{2i})\}}, \tag{3}$$

where the variance is replaced by the determinant of the lower-right $k \times k$ sub-matrix for $k > 1$, which bears some similarity to the idea behind D-optimal design of experiments. Clearly when a design has common variance, $r_{ACV} = 1$. When a design does not have common variance, r_{ACV} gives us an idea of how far we are from common variance. For example, if $r_{ACV} = 0.5$ then we know that between any two models under consideration, the largest variance of interaction terms is twice that of the smallest. This knowledge can hopefully help inform model selection.

Any off-the-shelf optimization algorithm could be used to try to maximize this objective function. We have chosen to use a genetic algorithm, as is common in the design literature (Mandal et al., 2015; Lin et al., 2015). Genetic algorithms are optimization techniques mimicking Darwin's idea of natural selection and survival of the fittest. This search expects that a good candidate solution will provide good offspring and imitates the way that chromosomes crossover and mutate when reproducing. Here, each chromosome is a design, and the fitness of a chromosome is determined by the corresponding objective function value. At each iteration the worst chromosomes are replaced with offspring generated by combining the settings from two better chromosomes, along with some small probability of a mutation. In the context of our problem, a mutation corresponds to randomly changing the settings for one of the factors in one of the runs. The algorithm terminates when either the maximum number of iterations has been reached, or a design with common variance has been found. The steps in our genetic algorithm are outlined in Algorithm 1.

The genetic algorithm requires the user to specify the mutation probability and the maximum number of iterations. Our experience with the algorithm suggests using a small mutation probability to encourage only one or two mutations each time a new chromosome is created. For our purposes, we generally use a maximum of 1000 iterations, although the algorithm is quite fast and this number can easily be increased if needed. Our algorithm is implemented in Julia version 1.0.2 and is available for download from the author's website.

Algorithm 1 Pseudo-code for the genetic algorithm to find A-ComVar designs.

```
function A-ComVarDesign(design problem, mutation prob., max iter., φ)
    for Each chromosome do
        initialize chromosome to random design
        Calculate fitness
    end for
    while termination criteria not met do
        Identify worst two chromosomes
        Use a crossover to generate two new chromosomes
        Mutate the two new chromosomes
        Replace the worst chromosomes with the two new chromosomes
        Calculate fitness for new chromosomes
    end while
end function
```

5 Numerical Experiments

5.1 Simulation 1 - Designs with One 2-Factor Interaction

We conducted a series of simulations to investigate the ability of our approach to find A-ComVar designs and to gain a better understanding of when common variance designs can be found. We started by examining designs with a single 2-factor interaction. We consider 2^{m_1} and 3^{m_2} experiments, with $m_1 = 4, \dots, 9$ and $m_2 = 3, \dots, 6$. For the 2^{m_1} experiments, we considered run sizes of $n_{m_1} = m_1 + 2, \dots, m_1 + 11$, and for the 3^{m_2} experiments, we considered run sizes of $n_{m_2} = 2m_1 + 2, \dots, 2m_1 + 11$. For each combination of settings, we ran our genetic algorithm 1000 times and stored the r_{ACV} results. The tuning parameters used were a mutation probability of 0.05 and a maximum of 1000 iterations.

Figure 1 displays the results for the 2^{m_1} simulations and Figure 2 displays the results for the 3^{m_2} simulations. We first note that our results are consistent with the findings of Ghosh and Chowdhury

(2017), who used exhaustive searches to identify common variance designs. For example, Ghosh and Chowdhury (2017) found that common variance designs exist for 3^3 designs with 8 runs, which agrees with the boxplots in the first panel of Figure 2. This supports our use of the genetic algorithm approach with the objective function described above. Furthermore, in cases where the common variance designs either do not exist or could not be found, our approach was able to find designs that attempt to get as close as possible to common variance. For example, it is known from exhaustive searches that no common variance design exists for a 3^3 experiment with 12 factors. However, the proposed approach was able to find designs where the smallest variance was greater than 0.8 times the largest variance, indicating that the design is quite close to having the common variance property.

5.2 Simulation 2 - Designs with Two 2-Factor Interactions

For designs with multiple 2-factor interactions (i.e. $k > 1$), we generalize the objective function in (2) by replacing $var(\hat{\beta}_{2i})$ with the determinant of the block of the variance covariance matrix corresponding to the interactions terms. That is, we take the determinant of the bottom-right $k \times k$ sub-matrix of $var(\hat{\beta})$.

To demonstrate the approach, we conducted another simulation with two 2-factor interactions (i.e. $k = 2$). We consider 2^{m_3} experiments, with $m_3 = 4, \ldots, 7$. We considered run sizes of $n_{m_1} = m_3 + 6, \ldots, m_3 + 12$. For each combination of settings, we ran our genetic algorithm 1000 times and stored the r_{ACV} results. The tuning parameters used were a mutation probability of 0.05 and a maximum of 1000 iterations.

Figure 3 shows the results for these simulations. As before, we can see that in many cases the genetic algorithm is able to find common variance designs. In cases where common variance designs cannot be found, the approach is often able to identify a design resulting in relatively close to common variance.

5.3 Simulation 3 - Comparison to Plackett-Burman Designs

Next, we examine the performance of two A-ComVar designs that do not achieve the common variance property, as well as one common variance design for $k = 1$ obtained from simulation. We perform model selection using the adaptive lasso regression technique as described earlier. We

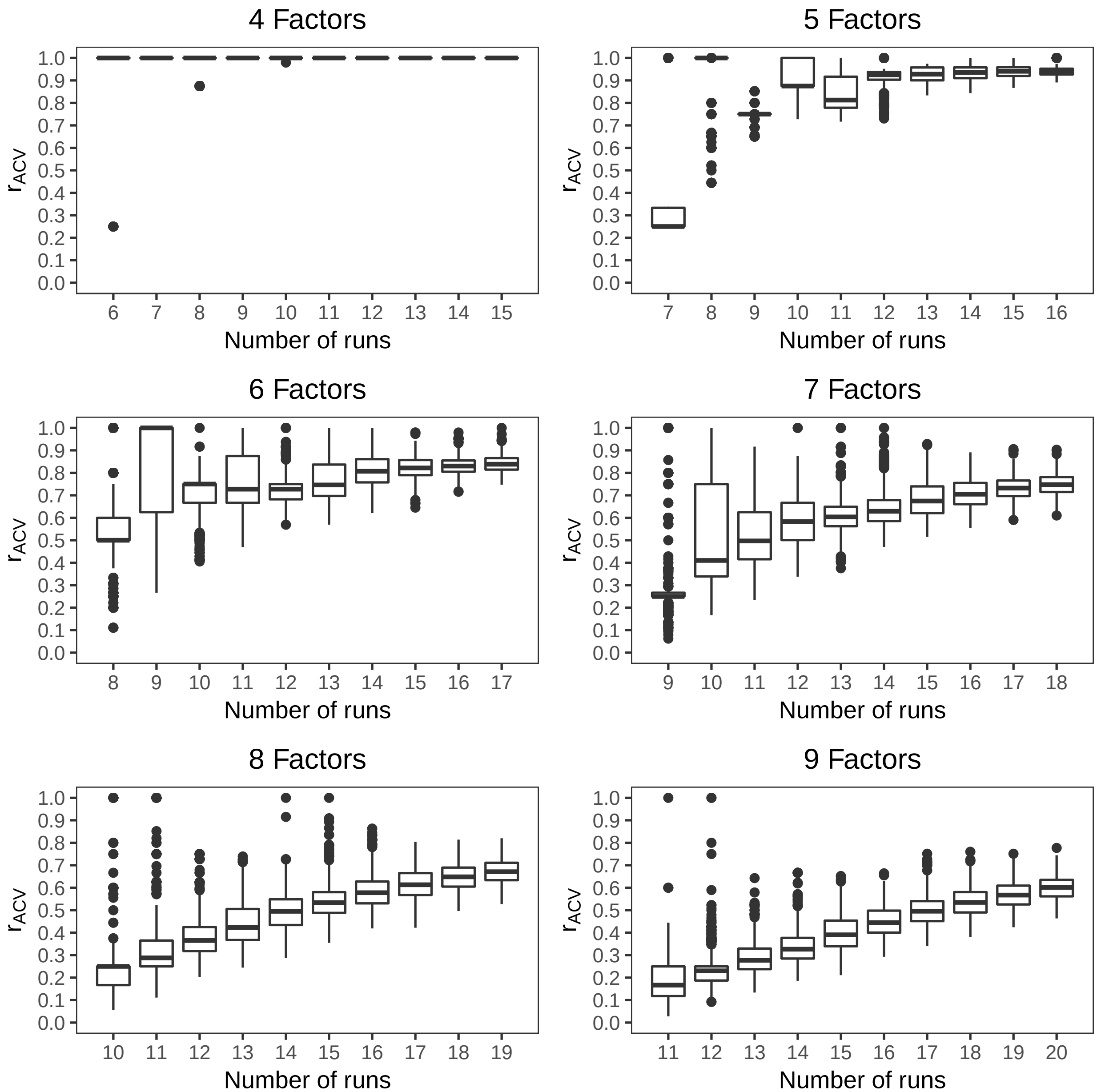

Figure 1: Ratios r_{ACV} for the 2^{m_1} simulations across 1000 replicates for each experimental setting.

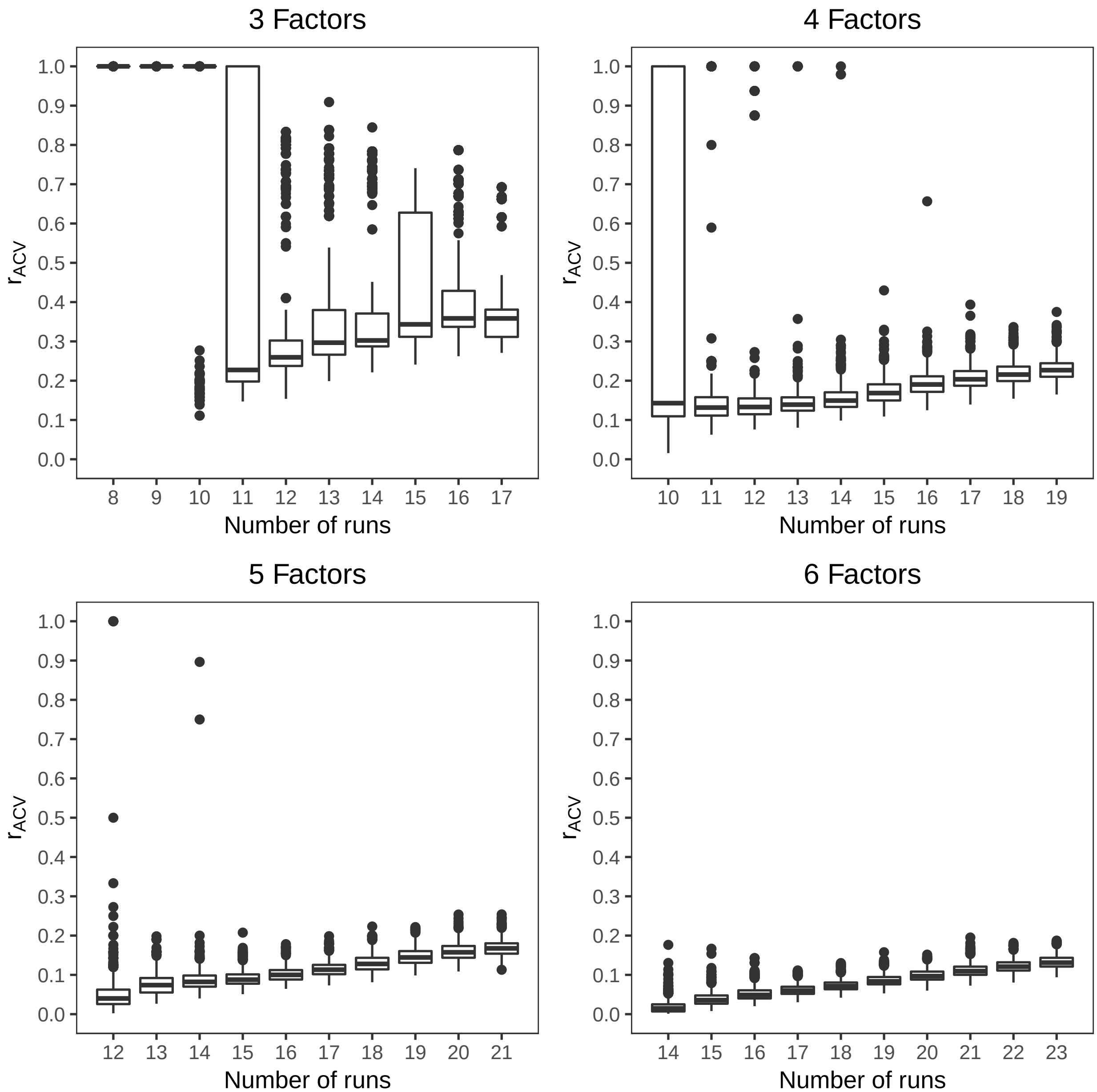

Figure 2: Ratios r_{ACV} for the 3^{m_2} simulations across 1000 replicates for each experimental setting.

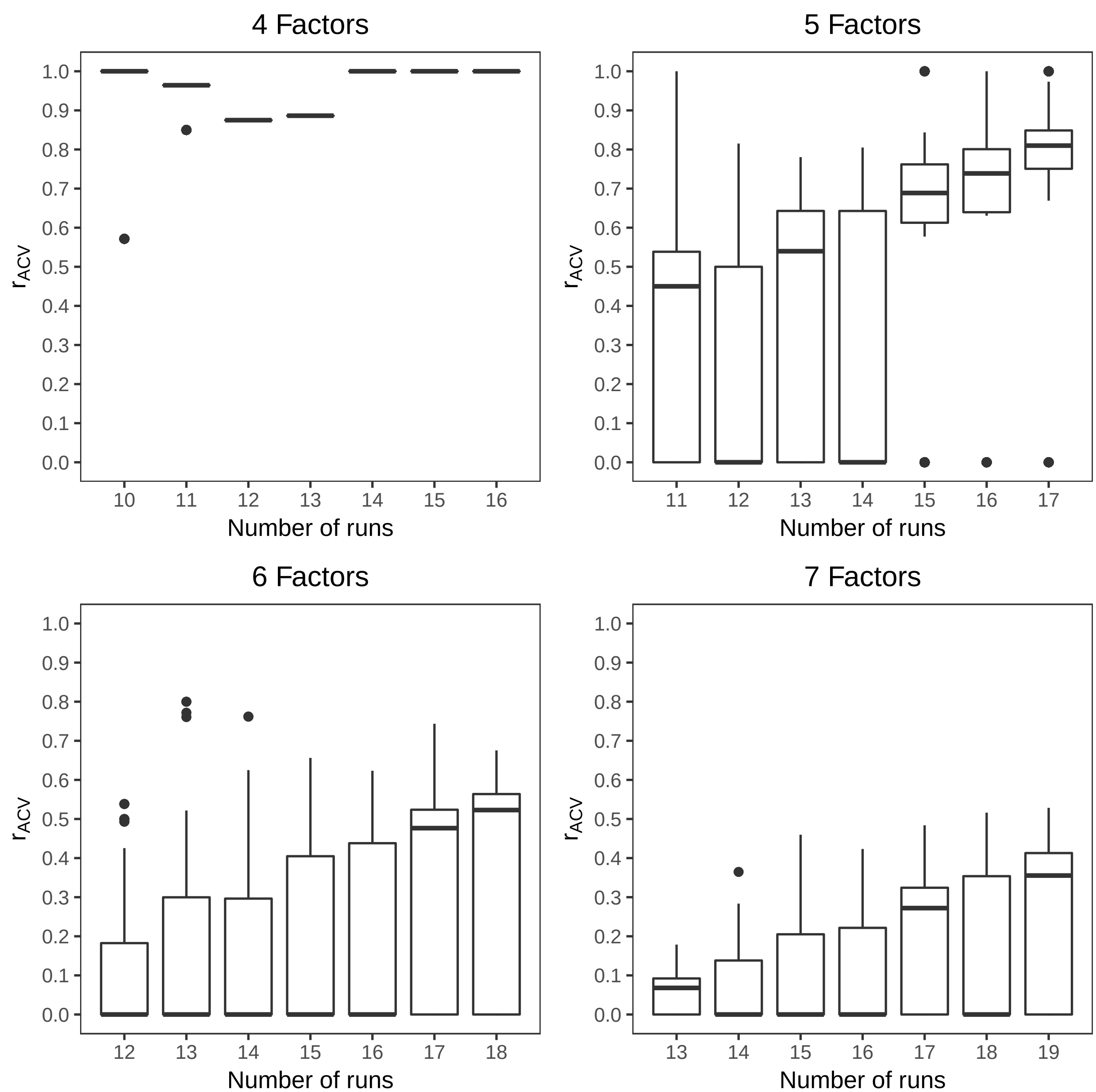

Figure 3: Ratios r_{ACV} for the 2^{m_1} simulations across 1000 replicates for each experimental setting with $k = 2$.

A	B	C	D	E
-1	-1	-1	-1	-1
-1	-1	-1	-1	1
-1	-1	-1	1	-1
-1	-1	1	1	1
-1	1	-1	-1	-1
-1	1	1	-1	-1
1	-1	-1	1	1
1	-1	1	1	1
1	1	-1	-1	-1
1	1	1	-1	1
1	1	1	1	-1
1	1	1	1	1

A	B	C	D
-1	-1	-1	-1
-1	-1	-1	1
-1	-1	1	1
-1	1	-1	-1
-1	1	1	-1
-1	1	1	1
1	-1	-1	-1
1	-1	-1	1
1	-1	1	-1
1	1	-1	-1
1	1	-1	1
1	1	1	1

A	B	C	D	E
-1	-1	-1	-1	-1
-1	-1	1	1	1
-1	1	-1	1	-1
-1	1	-1	1	1
-1	1	1	-1	1
-1	1	1	1	-1
1	-1	-1	1	1
1	-1	1	-1	1
1	-1	1	1	-1
1	1	-1	-1	1
1	1	-1	1	-1
1	1	1	-1	-1

Table 6: Three designs (1) D^1(left): Common variance design for $k = 1$ with $m = 5$ and $n = 12$, (2) D^2 (middle): Approximate common variance design for $k = 2$ with $m = 4$ and $n = 12$ and (3) D^3 (right): Approximate common variance design for $k = 2$, $m = 5$, $n = 12$, used in Simulation 3. For D^2 and D^3 the ratio of the minimum to the maximum variance is 0.87 and 0.81 respectively.

compare performance of each of the three designs D^1, D^2 and D^3 with the design having five factors $A - E$ from Plackett-Burman design presented in Table 2, using simulations. These three designs are presented in Table 5. D^1 is a common variance design for $k = 1$ with $m = 5$ and $n = 12$ found using our approach. D^2 is an A-ComVar design for $k = 2$ with $m = 4$ and $n = 12$. D^3 (right) is an A-ComVar design for $k = 2$, $m = 5$, $n = 12$. For D^2 and D^3 the ratio of the minimum to the maximum variance is 0.87 and 0.81 respectively.

We consider similar simulation setting as described in Section 3. We generate the data in simulation using the three designs presented in Table 5 and using the coefficients (both “big” and “small”) as described in Section 3. Tables 7-9 display the percentage of times the correct model is identified by the designs D^1-D^3, respectively, and the Plackett-Burman designs with factors $A - E$ under different factorial effect size combinations and varied choices of variance (σ^2) of the noise. On the whole, in each case designs obtained using our proposed approach perform comparably with the Plackett-Burman design, with the better design often depending on the amount of variability in the model in combination with the true underlying model.

Table 7: Percentage of Correctly Identified Models for common variance design D^1 and Plackett-Burman design with factors $A-E$.

	Model	Size	0.1		0.25		0.5		0.75		1	
			CV	PB	CV	PB	CV	PB	CV	PB	CV	PB
1	F_1	B	100	100	0	0	30	19	6	9	6	3
2	F_1	S	71	75	0	0	7	8	97	100	1	0
3	F_1+F_2	$B+B$	62	65	100	100	3	6	40	57	1	0
4	F_1+F_2	$B+S$	28	28	92	91	0	3	2	7	96	72
5	F_1+F_2	$S+S$	59	58	66	64	72	94	1	1	17	5
6	$F_1+F_1F_2$	$B+B$	89	84	52	50	74	60	0	0	5	0
7	$F_1+F_1F_2$	$B+S$	26	24	22	19	4	5	37	72	5	1
8	$F_1+F_1F_2$	$S+S$	16	12	96	94	13	9	4	15	0	0
9	$F_1+F_2+F_1F_2$	$B+B+B$	2	5	44	43	3	2	1	1	18	1
10	$F_1+F_2+F_1F_2$	$B+B+S$	8	3	6	7	93	93	0	0	4	0
11	$F_1+F_2+F_1F_2$	$B+S+B$	100	100	4	6	21	21	1	0	1	0
12	$F_1+F_2+F_1F_2$	$B+S+S$	70	75	7	0	0	0	42	54	0	0
13	$F_1+F_2+F_1F_2$	$S+S+S$	69	64	85	79	1	0	2	5	1	0
14	$F_1+F_2+F_3$	$B+B+B$	17	23	36	43	0	0	0	2	43	2
15	$F_1+F_2+F_3$	$B+B+S$	38	39	4	2	86	96	0	1	13	2
16	$F_1+F_2+F_3$	$B+S+S$	54	59	3	1	2	4	0	1	1	0
17	$F_1+F_2+F_3$	$S+S+S$	28	33	1	0	8	4	100	71	0	0
18	$F_1+F_2+F_1F_3$	$B+B+B$	3	13	100	100	0	0	87	43	0	1
19	$F_1+F_2+F_1F_3$	$B+B+S$	2	0	76	85	0	0	45	26	44	17
20	$F_1+F_2+F_1F_3$	$B+S+B$	2	2	42	55	100	99	8	1	1	0
21	$F_1+F_2+F_1F_3$	$B+S+S$	89	88	26	34	76	81	4	1	0	0
22	$F_1+F_2+F_1F_3$	$S+B+S$	19	30	23	19	35	43	98	55	0	0
23	$F_1+F_2+F_1F_3$	$S+S+S$	0	1	10	4	12	12	16	3	0	0

Table 8: Percentage of Correctly Identified Models for A-ComVar design D^2 and Plackett-Burman design with factors $A - E$.

	Model	Size	0.1		0.25		0.5		0.75		1	
			A-ComVar	PB	A-ComVar	PB	A-ComVar	PB	A-ComVar	PB	A-ComVar	PB
1	F_1	B	100	99	0	0	41	26	1	3	7	1
2	F_1	S	85	82	1	0	25	21	98	100	3	1
3	$F_1 + F_2$	$B + B$	55	65	100	100	7	4	54	61	0	0
4	$F_1 + F_2$	$B + S$	28	34	92	84	0	0	4	1	90	52
5	$F_1 + F_2$	$S + S$	41	42	59	58	29	49	0	0	7	5
6	$F_1 + F_1F_2$	$B + B$	59	60	37	41	10	6	1	0	4	1
7	$F_1 + F_1F_2$	$B + S$	22	26	30	27	6	2	56	93	0	1
8	$F_1 + F_1F_2$	$S + S$	19	18	88	93	1	1	0	1	1	0
9	$F_1 + F_2 + F_1F_2$	$B + B + B$	8	10	60	51	3	2	0	2	27	3
10	$F_1 + F_2 + F_1F_2$	$B + B + S$	5	2	17	10	85	86	0	0	6	2
11	$F_1 + F_2 + F_1F_2$	$B + S + B$	100	100	12	7	12	10	0	0	1	1
12	$F_1 + F_2 + F_1F_2$	$B + S + S$	79	73	2	2	5	1	42	68	0	0
13	$F_1 + F_2 + F_1F_2$	$S + S + S$	40	34	67	70	0	0	1	4	0	0
14	$F_1 + F_2 + F_3$	$B + B + B$	15	18	14	9	0	0	0	0	30	3
15	$F_1 + F_2 + F_3$	$B + B + S$	5	14	10	4	91	90	0	0	7	0
16	$F_1 + F_2 + F_3$	$B + S + S$	69	71	4	1	18	10	0	0	3	0
17	$F_1 + F_2 + F_3$	$S + S + S$	51	53	2	0	0	0	100	98	0	0
18	$F_1 + F_2 + F_1F_3$	$B + B + B$	3	4	100	100	0	0	67	33	0	1
19	$F_1 + F_2 + F_1F_3$	$B + B + S$	0	5	96	95	0	0	37	18	66	9
20	$F_1 + F_2 + F_1F_3$	$B + S + B$	3	0	20	32	100	100	12	4	1	0
21	$F_1 + F_2 + F_1F_3$	$B + S + S$	57	52	15	17	74	84	8	3	0	0
22	$F_1 + F_2 + F_1F_3$	$S + B + S$	21	27	29	27	17	23	100	98	0	0
23	$F_1 + F_2 + F_1F_3$	$S + S + S$	0	3	98	94	5	20	8	1	0	0

Table 9: Percentage of Correctly Identified Models for A-ComVar design D^3 and Plackett-Burman design with factors $A - E$.

	Model	Size	0.1		0.25		0.5		0.75		1	
			A-ComVar	PB	A-ComVar	PB	A-ComVar	PB	A-ComVar	PB	A-ComVar	PB
1	F_1	B	99	99	1	1	38	24	7	12	5	0
2	F_1	S	89	92	0	0	9	4	89	91	0	0
3	$F_1 + F_2$	$B + B$	65	76	100	100	11	8	6	11	0	1
4	$F_1 + F_2$	$B + S$	50	40	89	88	2	4	0	3	97	81
5	$F_1 + F_2$	$S + S$	60	65	55	65	98	100	0	5	45	23
6	$F_1 + F_1F_2$	$B + B$	85	83	50	46	36	47	1	3	8	3
7	$F_1 + F_1F_2$	$B + S$	19	33	40	30	8	8	60	91	2	1
8	$F_1 + F_1F_2$	$S + S$	11	10	25	19	4	2	1	8	3	0
9	$F_1 + F_2 + F_1F_2$	$B + B + B$	6	13	11	19	2	8	0	0	65	16
10	$F_1 + F_2 + F_1F_2$	$B + B + S$	9	4	5	10	66	47	0	1	11	2
11	$F_1 + F_2 + F_1F_2$	$B + S + B$	100	100	2	3	27	7	0	1	0	0
12	$F_1 + F_2 + F_1F_2$	$B + S + S$	73	76	6	1	3	0	64	85	0	0
13	$F_1 + F_2 + F_1F_2$	$S + S + S$	25	33	89	89	2	0	2	10	0	0
14	$F_1 + F_2 + F_3$	$B + B + B$	27	25	22	33	0	1	0	0	90	30
15	$F_1 + F_2 + F_3$	$B + B + S$	26	30	21	7	81	87	0	0	5	0
16	$F_1 + F_2 + F_3$	$B + S + S$	99	100	3	4	14	13	0	0	0	2
17	$F_1 + F_2 + F_3$	$S + S + S$	5	14	1	1	1	0	100	37	0	0
18	$F_1 + F_2 + F_1F_3$	$B + B + B$	5	12	100	100	1	0	83	30	0	0
19	$F_1 + F_2 + F_1F_3$	$B + B + S$	3	2	78	92	0	0	38	6	49	25
20	$F_1 + F_2 + F_1F_3$	$B + S + B$	3	3	25	33	100	100	16	6	7	4
21	$F_1 + F_2 + F_1F_3$	$B + S + S$	63	60	34	34	42	69	4	0	0	0
22	$F_1 + F_2 + F_1F_3$	$S + B + S$	5	2	6	7	24	24	98	87	0	1
23	$F_1 + F_2 + F_1F_3$	$S + S + S$	1	2	100	100	6	19	43	19	0	0

6 Discussion

In this work we introduced A-ComVar Designs, an extension of common variance designs. These designs address the difficulties associated for finding common variance designs via exhaustive search and allow a relaxation of the common variance property for cases where a common variance design does not exist. Through simulation, we demonstrated that the proposed algorithmic approach allows us to quickly find common variance designs that overlap with those known in the literature. Furthermore, in cases where common variance designs do not exist or cannot be found, our approach allows identification of designs with close to common variance. Comparisons to Plackett-Burman designs demonstrated that such designs perform quite well in practice, and that in many cases these A-ComVar designs were as good as common variance designs.

There are several avenues here for future work. First, we considered only cases with 2-level and 3-level factors. Future work could consider finding A-ComVar designs with a mixture of both. Second, we utilized a genetic algorithm to find these designs. There are numerous other optimization approaches that could be used to maximize the objective function in (2). In some cases, these other approaches may succeed in finding designs with a much better ratio of minimum to maximum variance of the uncommon parameters. Third, there is another approach to finding common variance designs through hierarchical designs (Chowdhury, 2016). These designs are found by identifying a common variance design for a smaller number of runs and then adding runs while trying to preserve the common variance property. It is possible that a similar idea could be developed for A-ComVar designs. Finally, future work could study the types of A-ComVar designs that can be found when the number of interactions in the model increases beyond two.

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On behalf of all authors, the corresponding author states that there is no conflict of interest.