2.6 Exercises

1. For semivariogram models 2, 4, 5, 6, 7, and 8 in Subsection 2.1.3,
(a) identify the nugget, sill, and range (or effective range) for each;
(b) find the covariance function \( C(t) \) corresponding to each \( \gamma(t) \), provided
it exists.

2. Prove that for Gaussian processes, strong stationarity is equivalent to
weak stationarity.

3. Consider the triangular (or "tent") covariance function,
\[
C_h(x) = \begin{cases} \sigma^2(1 - |x|/a) & \text{if } |x| \leq a, \\ \sigma^2 & \text{if } |x| > a, \\ 0 & \text{if } |x| \leq 0. \end{cases}
\]

It is valid in one dimension. (The reader can verify that it is the char-
acteristic function of the density function \( f(x) \) proportional to \([1 - |x|]/\sigma^2 \).)
Now in two dimensions, consider a \( 6 \times 6 \) grid with \( k =
(\sqrt{5}/\sqrt{2}), j = 1, \ldots, 6, k = 1, \ldots, 8 \). Assign \( \sigma_k \) to
\( \sigma_k \) such that \( \sigma_k = 1 \) if \( j+k \) is even, \( \sigma_k = 1 \) if \( j+k \) is odd. Show that
\( \text{Var} [X_{ij} | Y_{ij}] < 0 \), and hence that the triangular covariance function
is invalid in two dimensions.

4. The turning bands method (Christakos, 1984; Strahler, 1999) is a technique
for creating stationary covariance functions on \( \mathbb{R}^n \). Let \( u \) be a random
unit vector on \( \mathbb{R}^n \) (by random we mean that the coordinate vector that
defines \( u \) is randomly chosen on the surface of the unit sphere in \( \mathbb{R}^n \)),
let \( \xi(u) \) be a valid stationary covariance function on \( \mathbb{R}^n \), and let \( W(t) \)
for any location \( s \) in \( \mathbb{R}^n \), define
\[
Y(s) = W(u^t s).
\]

Note that we can think of the process either conditionally given \( u \) or
unconditionally. Note also that \( Y(s) \) has the possibly undesirable property that it is
correlated on planes \( \{ s \in \mathbb{R}^n | u^t s = k \} \).

(a) If \( W \) is a Gaussian process, show that, given \( u, Y(s) \) is also a Gaussian
process and is stationary.

(b) Show that \( Y(s) \) is not a Gaussian process, but is isotropic.

Hint: Show that \( C_n(Y(u), Y(u^t s)) = \text{Cov}(s, s^t u) \).

5. (a) Based on (2.16), show that \( G(0) \) is a valid correlation function; i.e.,
that \( G \) is a bounded, positive, symmetric about 0 measure on \( \mathbb{R}^2 \).

(b) Show further that if \( G \) and \( \gamma \) are isotropic, then \( G(0) = \gamma(0) \).

6. What is the issue with regard to specifying \( \gamma(t) \) in the covariance
function estimate (2.14)?

Exercises

(b) Show either algebraically or numerically that regardless of how \( \gamma(t) \)
is obtained, \( \gamma(t) \neq \gamma(0) - \gamma(t) \) for all \( t \).

7. Carry out the steps outlined in Section 2.3.1 in \texttt{SpatialStats}. In
addition:

(a) Provide a descriptive summary of the scallop data with the plots
derived from the above session.

(b) Experiment with the \texttt{model.variogram} function to obtain rough es-
imates of the nugget, sill, and range; your final objective function
should have a value less than 9.

(c) Repeat the theoretical variogram fitting with an exponential variogram,
and report your results.

8. Consider the coal.ash data frame built into \texttt{SpatialStats}. This data
comes from the Pittsburgh coal seam on the Robinson Mine property in
Greene County, PA (Cressie, 1993, p. 32) This data frame contains 208
coal ash core samples (the variable coal in the data frame) collected on
a grid given by \( x \) and \( y \) planar coordinates (not latitude and longitude).

Carry out the following tasks in \texttt{Splus}:

(a) Plot the sampled sites embedded on a map of the region. Add contour
lines to the plot.

(b) Provide a descriptive summary (histograms, stems, quartiles, means, range,
etc.) of the variable coal in the data frame.

(c) Plot variograms and correlograms of \( X \) response and comment on the need
for spatial analysis here.

(d) If you think that there is need for spatial analysis, use the interactive
\texttt{model.variogram} method in \texttt{Splus} to arrive at your best estimates of
the range, nugget, and sill. Report your values of the objective functions.

(e) Try to estimate the above parameters using the \texttt{nls} procedure in
\texttt{Splus}.

HINT: You may wish to look at Section 32 in Kahanuz et al. (1998) for
some insight into the coal.ash data.

9. Confirm expressions (2.18) and (2.19), and subsequently verify the form
for \( \Lambda \) given in equation (2.20).

10. Show that when using (2.18) to predict the value of the surface at one of
the existing data locations \( s_k \), the predictor will equal the observed value
at that location if and only if \( r^2 = 0 \). (That is, in the usual Gaussian process
a spatial interpolator only in the "no-where prediction" scenario.)

11. It is an unfortunate feature of \texttt{SpatialStats} that there is no intrinsic