1. Consider the regression model

\[ y_i = \beta_0 + \beta_1 x_i + \beta_2 (x_i^2 - 5)/4 + \varepsilon_i, \quad i = 1, \ldots, 4, \]

where \( \varepsilon_1, \ldots, \varepsilon_4 \) are i.i.d. with \( \text{E}(\varepsilon_i) = 0, \text{var}(\varepsilon_i) = \sigma^2 \). Suppose \( \mathbf{x} = (x_1, \ldots, x_4)^T = (-3, -1, 1, 3)^T \), and \( \mathbf{y} = (y_1, \ldots, y_4)^T = (1, 2, 2, 4)^T \).

a. (6 points) Find the BLUE of \( \mathbf{\beta} = (\beta_0, \beta_1, \beta_2)^T \) from this model.
b. (8 points) Based on this model, obtain a 95% CI for $E(y)$ when $x = 0$. 
*Hint:* MSE = .45 in this problem.
2. (10 points) Let $y_i = \beta x_i + \varepsilon_i$, $i = 1, 2$, where $\varepsilon_1 \sim N(0, \sigma^2)$ and $\varepsilon_2 \sim N(0, 2\sigma^2)$, and $\varepsilon_1$ and $\varepsilon_2$ are statistically independent. If $x_1 = 1$ and $x_2 = -1$, then obtain the BLUE of $\beta$ and find the variance of your estimator.
3. (12 points) In homework # 5, problem # 8.19, you used Lagrange multipliers to show that the least-squares estimator of $\beta$ in the full-rank model

$$ y = X\beta + \varepsilon, \quad \text{subject to the constraint } C\beta = 0, $$

where $E(\varepsilon) = 0$, and $\text{var}(\varepsilon) = \sigma^2 I$, is

$$ \hat{\beta}_c = \hat{\beta} - (X^TX)^{-1}C^T\{C(X^TX)^{-1}C^T\}^{-1}C\hat{\beta}, $$

where $\hat{\beta}$ is the unconstrained least squares estimator of $\beta$. Use a different method (don’t use Lagrange multipliers) to prove this result.

**Hint:** Use the facts that 1) $\beta = (X^TX)^{-1}X^T\mu$ and 2) subject to the constraint, the model space is $V_0 = C(T)^\perp \cap C(X)$, where $T = X(X^TX)^{-1}C^T$. 


4. Recall the chemical reaction data that you worked on in problem # 8.41 of homework # 5. For these data, $y =$percent of unchanged starting material, $x_1 =$temperature, $x_2 =$concentration of a reagent, and $x_3 =$time of reaction. I fit the model

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \varepsilon_i, \quad i = 1, \ldots, n = 19,$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^T \sim N(0, \sigma^2 I)$. I obtained the following results:

$$\hat{\beta} = \begin{pmatrix} 332.11 \\ -1.55 \\ -1.42 \\ -2.24 \end{pmatrix} \quad \text{vár}(\hat{\beta}) = \begin{pmatrix} 349.43 & -1.81 & -1.67 & -.11 \\ -1.81 & .0098 & .0068 & -.0023 \\ -1.67 & .0068 & .022 & -.0094 \\ -.11 & -.0023 & -.0094 & .12 \end{pmatrix}$$

and $s^2 = \text{MSE} = 5.34, R^2 = .96$.

a. (6 points) Compute the $F$ test statistic for the hypothesis $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$, and give the reference distribution against which this test statistic should be compared for a test of $H_0$.

b. (5 points) Express the hypothesis $H_0 : 2\beta_1 = \beta_3 = -3$ in the form of the general linear hypothesis.
c. (8 points) Compute the $F$ test statistic for $H_0$ from part (b) and give the reference distribution for this test statistic.