The Prym Torelli problem: an update and a reformulation as a question in birational geometry

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§0. Introduction: We are concerned with an open question, the precise classical Prym Torelli problem: which unramified double covers of smooth curves are uniquely determined by their Prym varieties? Since a Prym variety for which Torelli is true can be identified with a unique double cover, every intrinsic property of the double cover corresponds to an intrinsic property of the Prym variety. In particular the theta divisor Ξ on such a Prym variety has a distinguished Abel parametrization $\varphi: X \to \Xi$ defined by projectivizing any one of a distinguished family of reflexive rank two sheaves. Hence resolution of the Torelli problem would provide a useful tool for understanding these interesting principally polarized abelian varieties (ppav's) and their theta divisors. This paper is a report on the current status of attempts to generalize to the Prym setting, two famous approaches to the Torelli problem for Jacobians, the "base locus of quadric tangent cones" method of Andreotti-Mayer and Green [AM, M2, G], and the "branch divisor of the Gauss map" method of Andreotti [A], both of which refer to constructions on the theta divisor. We will also describe for Prym varieties the "infinitesimal variation of Hodge structures" (IVHS) approach to Torelli problems of Carlson and Griffiths [CG], and an analog, Thm.5.9, of a result of Kempf [Ke1, p.16; Ke2, Cor.4.4, p.253] linking the Torelli problem to properties of "Picard sheaves", i.e. (higher) direct images of Poincare bundles on abelian varieties. Although the article's goal is primarily expository, much of the material discussed is very recent [SV5, SV6, SV7], some arguments (such as the IVHS argument 3.4(i) for degree one Torelli for Pryms) seem not to have occurred in print before and some results (such as the density of double points in the stable singular locus of Prym theta divisors Thm.3.5, and the requisite Riemann singularities theorem for double points which are both stable and exceptional, prop.3.6) are new. The recovery of a curve of genus g from its symmetric product $C^{(g+1)}$ (and a cohomology class), Prop.5.8.ff., is similar in spirit to the result of Kempf above. It differs in detail from that result in the sense that it uses the generically rank 2 sheaf $\pi_*\mathcal{L}$ derived from a Poincare line bundle $\mathcal{L}$ in degree g+1, as opposed to Kempf's use of the rank d+g-1 vector bundle $R^1\pi_*\mathcal{L}$ derived from a Poincare line bundle $\mathcal{L}$ of negative degree -d. Since the fibers of $\pi_*(\mathcal{L})$ at general points L in $Pic^{g+1}(C)$, are spaces of sections of L, this sheaf can be approached more geometrically than $R^1\pi_*\mathcal{L}$. We work over the complex numbers.

Acknowledgements: This paper is dedicated to C. H. Clemens, from whom we both learned about Prym varieties and many other beautiful things, on his 60th birthday. Our hope is to provide a successor to the excellent surveys [Cl, Be4].

§1. The Prym Torelli problem and Donagi's conjecture

Definition 1.1. Prym varieties and the Prym map.

Consider a connected étale double cover $\pi:\tilde{C}\to C$ of a smooth non hyperelliptic curve C of genus $g \geq 3$, and the norm map $Nm:Pic^0(\tilde{C})\to Pic^0(C)$ on line bundles it determines. According to [M2, bottom p.329, where Nm = ψ, cf. (b), bottom p.341] the inverse image of 0 by this surjective map has two connected components $Nm^{-1}(0) = P_0 \cup P_1$. Denote by P_0 the component containing 0, and by P_1 the component not containing 0. The principal polarization of the Jacobian of $\tilde{C}$, considered as a cohomology class $\tilde{\vartheta}$ in $H^2(\mathrm{Pic}^0(\tilde{C}),\mathbb{Z})$, has restriction $\tilde{\vartheta}|P_0 = 2\xi$, to P_0, equal to twice a principal polarization ξ on P_0. The resulting pair (P_0,ξ) is by definition the principally polarized Prym variety determined by π. If C has genus g, the Riemann Hurwitz formula implies $\tilde{C}$ has genus 2g-1, hence P_0 has dimension $p = g-1$. Setting $\mathcal{P}(\tilde{C}/C) = (P_0,\xi)$ defines a morphism $\mathcal{P}:\mathcal{R}_g \to \mathcal{A}_{g-1}$, the Prym map, from the moduli space of double covers of curves of genus g, to the moduli space of principally polarized abelian varieties of dimension $p = g-1$.

Statement of the Prym Torelli problem.

Problem 1.2. Define $\mathcal{V} \subset \mathcal{A}_{g-1}$ to be the largest open set such that the restriction $\mathcal{P}|\mathcal{P}^{-1}(\mathcal{V})$ of the Prym map $\mathcal{P}$ over $\mathcal{V}$ is injective, and put $\mathcal{U} = \mathcal{P}^{-1}(\mathcal{V})$.

(i) Describe the subset $\mathcal{U} \subset \mathcal{R}_g$.

(ii) Define an inverse map $\mathcal{V} \to \mathcal{U}$.

Terminology: Part (i) will be called the "precise" Prym Torelli problem, and part (ii) the "precise constructive" Prym Torelli problem.

Currently one knows the following "generic" results on the Prym Torelli problem.

Theorem 1.3.

(i) If $g \geq 7$ then $\mathcal{U} \neq \emptyset$; in particular the Prym map is injective on some non empty open subset of $\mathcal{R}_g$, ([FS], see also [Ka]).

(ii) If $g \geq 8$ there is a geometrically defined inverse for the restriction of $\mathcal{P}$ to some non empty open dense subset of $\mathcal{R}_g$, [We3, De2].

To the best of our knowledge, not a single double cover $\tilde{C}/C$ belonging to $\mathcal{U}$ is known. If we remove the restriction that $\mathcal{V}$ be open, then Debarre in [De1, Th.5.3, p.558; De3, Rmk.4.4.(2), p.515] has given a class of special tetragonal examples (at which the map $\mathcal{P}$ is ramified), and another class of doubly covered plane curve examples. If we replace "injective" by "quasi finite" in the definition of $\mathcal{U}$, then Wirtinger asserted [Wi, §59, pp.124-5] and Beauville proved [Be1, Cor.7.11] that $\mathcal{U} \neq \emptyset$ for $g \geq 6$. In this sense the problem is now over 100 years old. In the complementary direction it follows from works of Mumford [M2, Th.(a), p.344; cf. S, Th.(iii), p.212-3], Donagi [Do1], and Verra [V] that $\tilde{C}/C$ does not belong to $\mathcal{U}$ if C has (respectively) a g^1_2, a g^1_4 or a g^2_6, the last case being possible only for $g = 10$. The principal conjecture, which has guided the subject since proposed in almost this form by Donagi some 20 years ago [Do1], is that these are the only counterexamples to Prym Torelli. To state it, recall that $\mathrm{Cliff}(C) = \min.\{\deg(D) - 2\dim|D|$: D is a divisor on C with $h^0(D) \geq 2$, $h^1(D) \geq 2$, and $|D|$ the associated linear series$\}$.

Donagi's conjecture (modified by Verra and Lange-Sernesi).
Let $\mathcal{V} \subset \mathcal{A}_{g-1}$ (as above) be the largest open set such that the restriction $\mathcal{P}|\mathcal{P}^{-1}(\mathcal{V})$ of the Prym map $\mathcal{P}$ over $\mathcal{V}$ is injective, and $\mathcal{U} = \mathcal{P}^{-1}(\mathcal{V})$.

Conjecture 1.4. $\mathcal{U} = \{\tilde{C}/C$ in $\mathcal{R}_g$ such that $\mathrm{Cliff}(C) \geq 3\}$.

(See [Do1, Conj.4.1, p.183], [V, p.3], [LS, p.398]. Note the set of double covers $\{\tilde{C}/C$ in $\mathcal{R}_g$ with $\mathrm{Cliff}(C) \geq 3\}$ is non empty precisely for $g \geq 7$.)

With hindsight, Mumford's result [M2, Th.a)-d), p.344] that curves C with infinitely many g^1_4's have very special Prym varieties might have suggested that other curves with g^1_4's also define special Prym varieties. Donagi apparently discovered this phenomenon directly, from studying the branching of the Prym map for g = 6. This very insightful conjecture, completely unexpected when first proposed, is now supported by substantial evidence in the work of Debarre [De1, De2], Green - Lazarsfeld [GL], and Lange - Sernesi [LS]. Debarre in [De1, De3] adapts the method of Welters in [We3], while the other works generalize the approach proposed by Andreotti-Mayer ([M3]) and completed by Green [G, cf. SV2] for Jacobian varieties. Note that although every smooth curve is determined by its Jacobian variety, the AMG approach to constructing an inverse assumes Cliff(C) ≥ 2 (see however [BV], and [CS]). We recall this approach next.

§2. The "base locus of quadrics" method for Jacobians

2.1. Outline of the method. Denote by ω_C the very ample canonical line bundle of a curve C with g ≥ 3 and Cliff(C) ≥ 1, and by C_ω the corresponding canonical model of C in $|\omega_C|^* = \mathbb{P}^{g-1}$. If $(J(C), \vartheta)$ is the principally polarized Jacobian variety of C, the idea of this method is to recover the canonical model C_ω from the base locus of tangent cones at double points of the theta divisor $\Theta(C)$, (the unique divisor in J(C) (up to translation) with chern class ϑ = the polarizing class of J(C) in $H^2(J(C),\mathbb{Z})$). Let singΘ = the singular locus of Θ, sing$_2\Theta$ = the set of double points on Θ, $S^2H^0(\omega_C)$ = the vector space of quadratic polynomials on $\mathbb{P}^{g-1}$, $I_2(C_\omega)$ = the subvector space of quadratic polynomials vanishing on C_ω, $\mathbb{P}I_2(C_\omega)$ = the projective space of quadrics containing C_ω, and $\mathbb{P}(TC_p\Theta(C))$ = the projectivized tangent cone to Θ at p. Using the natural identification $|\omega_C|^* \cong \mathbb{P}(T_pJ)$, there are three ingredients to this proof of Torelli.

(i) Enriques' theorem [E]: If Cliff(C) ≥ 2 then C_ω is the unique curve in the base locus $\cap\{Q: Q \text{ in } \mathbb{P}I_2(C_\omega)\}$, of $\mathbb{P}I_2(C_\omega)$.

(ii) Riemann singularities package, [AM]: If Cliff(C) ≥ 1, then singΘ has pure dimension g-4, sing$_2\Theta$ is open and dense in singΘ, and there is a map γ:sing$_2\Theta(C) \rightarrow \mathbb{P}(I_2(C_\omega))$ taking p to $\mathbb{P}(TC_p\Theta(C))$. Moreover double points p such that γ(p) has rank 4 are dense in singΘ when g $\geq$ 5 [cf. SV2, Lemma 6, p.381].

(iii) Green's theorem [G]: If g(C) $\geq$ 5 and Cliff(C) $\geq$ 1, the image $\gamma(\text{sing}_2\Theta(C))$ of the AM map spans $\mathbb{P}(I_2(C_\omega))$, thus $\mathbb{P}(I_2(C_\omega))$ is spanned by tangent cones at rank 4 double points of Θ.

Cor. 2.2. [of (i), (ii), (iii)] If Cliff(C) $\geq$ 2, and g(C) $\geq$ 5, then Θ(C) determines C_ω as the unique curve in the base locus of the tangent cones at all rank 4 double points of Θ(C). (In fact C_ω equals this base locus, as a set [Ba], and as a scheme [P, cf. SD].)

Remarks 2.3

(i) The theorem in 2.1(iii) was proved by Andreotti - Mayer if g $\geq$ 5 and C is trigonal, hence for g $\geq$ 5 and C generic, [AM, Prop.4, p.199], and by Arbarello Harris for g = 6 and Cliff(C) $\geq$ 1 [AH]. A proof by deformation theory for C with Cliff(C) $\geq$ 1, g $\geq$ 5, over any algebraically closed field of characteristic $\neq$ 2 is in [SV2], and sketched below in §2.5.

(ii) The fact that "rank 4" double points are dense in sing$_2\Theta$(C) is technically useful in the proof of the spanning result, since it implies singΘ is generically reduced with its intrinsic scheme structure defined by an equation for Θ and its partials. The "Brill Noether" scheme structure [ACGH, p.177] on $W^1{}_{g-1}$ (which equals singΘ as a set) on the other hand, depends on a representation of the ppav (J,Θ) as a Jacobian of a specific curve, and can apparently differ from the intrinsic scheme structure on singΘ. For example Theorem (4.4), p.320 [ACGH], implies that the cycle structure of $W^1{}_3(C)$ is constant for all non hyperelliptic curves of genus 4, whereas the cycle underlying the intrinsic scheme singΘ in that case varies depending on whether or not C has a vanishing even theta null. The fact that analogous "Brill Noether" scheme structures for subsets of a Prym theta divisor Ξ [We2] are defined relevant to a choice of double cover, signals particular care in use of that structure when deducing results about Ξ, and is a primary reason to seek a precise Torelli theorem for Pryms.

2.4. Enriques' theorem implies generic Torelli for Jacobians. Although all three steps in 2.1 above are needed for a precise Torelli result, Enriques' theorem alone implies "generic Torelli" (injectivity of the Jacobi map on some unspecified non empty open set) by the IVHS or "infinitesimal variations of Hodge structure" method of Carlson-Griffiths [CG, Do2], as follows. They observe that a map $f:X \to Y$ is generically injective if and only if the derived map taking p to the image space $f_*(T_pX)$ is so, i.e. iff a general p in X is determined by the subspace $f_*(T_pX) \subset T_{f(p)}Y$. To apply this to Jacobians, one interprets the derivative of the Jacobi map in terms of the space $I_2(C_\omega)$ as follows. If $\mathcal{J}:\mathcal{M}_g \to \mathcal{A}_g$ is the Jacobi map from the moduli space of genus g curves to the moduli space of ppav's of dimension g, then the subspace $\mathcal{J}_*(T_C\mathcal{M}_g) \subset T_{\mathcal{J}(C)}\mathcal{A}_g$ determines its orthocomplement $(\mathcal{J}_*(T_C\mathcal{M}_g))^\perp$ in $(T_{\mathcal{J}(C)}\mathcal{A}_g)^*$. This inclusion in turn is equivalent to the inclusion $I_2(C_\omega) \subset S^2H^0(\omega_C)$, hence determines C_ω by Enriques' theorem. Thus $\mathcal{J}$ is generically injective. This argument has a generalization (see 3.4(i) below) to a proof of generic Torelli for Pryms. Unfortunately one does not know how to turn the precise statement of Enriques' theorem into a precise injectivity statement even for the Jacobi map. One attempt, for g = 4, is as follows. If $\mathcal{N}_{g-4} = \{(A,\Theta)$: $\dim.\text{sing}\Theta \geq g-4\} \subset \mathcal{A}_g$, knowing "generic Torelli" (and quasi finiteness) implies that the Jacobi map $\mathcal{J}:\mathcal{M}_g \to \mathcal{N}_{g-4}$ is essentially the normalization of one irreducible component Z of $\mathcal{N}_{g-4}$, hence $\mathcal{J}$ is injective over normal points of Z. Thus the problem is to identify which points of Z are normal points. When g = 4, the theory of "Milnor multiplicities" developed in [SV1, SV3] shows that $\mathcal{N}_0$, hence also Z, is normal at all Jacobians of non hyperelliptic chen C is non hyperelliptic, [Be3y two distinct $g^1{}_3$'s, i.e. at non hyperelliptic genus four curves with no vanishing even theta null. It is an open problem how to generalize this argument to higher genus Jacobians, and to Pryms.

2.5. A deformation theoretic proof of Green's theorem.

We sketch the idea of the proof in [SV2], in the hope of suggesting how it may be generalized to Prym varieties. An essential ingredient is a result of Kempf on deformations of symmetric products of curves [Ke3]. One shows, using the Riemann singularities package 2.1(ii) above, that every first order deformation of the non hyperelliptic curve C induces an "equisingular" first order deformation of $\Theta(C)$, and this correspondence can be identified with the derivative $\mathfrak{J}_*$ of the Jacobi map (on the level of functors). Hence if (to first order) all equisingular deformations of $\Theta(C)$ come from deformations of C, then $\Theta(C)$ determines $\mathfrak{J}_*(T_C\mathfrak{M}_g)$ which determines C by Enriques. Since for Cliff(C) ≥ 1 the Abel map $\alpha: C^{(g-1)} \to \Theta(C)$ is a "small" resolution of the singularities of $\Theta(C)$, all equisingular deformations of $\Theta(C)$ correspond to deformations of $C^{(g-1)}$, all of which come from deformations of C by Kempf's theorem [cf. SV2, bottom p.396]. This proves the result. Next we look at attempts to generalize the program 2.1 to Prym varieties.

§3. The base locus of quadrics method for Prym varieties

3.1. Prym canonical curves, singularities of Prym theta divisors, and Abel parametrizations.

If η is the unique non zero line bundle in the kernel of the map $\pi^*:\mathrm{Pic}^0(C) \to \mathrm{Pic}^0(\tilde{C})$ induced by the double cover $\pi:\tilde{C} \to C$, there is a "Prym canonical" model $C_\eta \subset |\omega_C \otimes \eta|^* = \mathbb{P}^{g-2}$ of the curve C defined by the line bundle $\omega_C \otimes \eta$ (which is very ample if Cliff(C) ≥ 3). Since the pair (C,η) determines the double cover $\pi:\tilde{C} \to C$, recovery of the embedded model C_η would determine the bundle $\omega_C \otimes \eta$, hence η and thus also the double cover, so would prove Prym Torelli.

Producing quadrics containing C_η from the Prym theta divisor Ξ, is more subtle than for Jacobians. In particular we must distinguish two types (not mutually exclusive) of singularities of Ξ associated to a given double cover $\tilde{C}\to C$ representing the polarized variety (P,Ξ). First of all a double cover $\pi:\tilde{C}\to C$ such that $(P(\tilde{C}/C),\Xi(\tilde{C}/C)) \cong (P,\Xi)$, defines an Abel parametrization $\varphi:X\to\Xi$ of the Prym theta divisor. I.e. if $g(C) = g$, $\tilde{g} = g(\tilde{C}) = 2g-1$, the double cover $\pi:\tilde{C}\to C$ induces a norm map $Nm:Pic^{2g-2}(\tilde{C})\to Pic^{2g-2}(C)$ and the inverse image $Nm^{-1}(\omega_C)$ has two connected components [M2, p.341-2] P and P^-, where $P = \{L: Nm(L) = \omega_C$ and $h^0(L)$ is even$\}$ and $P^- = \{L: Nm(L) = \omega_C$ and $h^0(L)$ is odd$\}$. Since $2g-2 = \tilde{g}-1$, the image of the Abel map $\tilde{\alpha}:\tilde{C}^{(2g-2)}\to Pic^{2g-2}(\tilde{C})$ is the natural model $\tilde{\Theta}\subset Pic^{2g-2}(\tilde{C})$ of the theta divisor for $\tilde{C}$, and by [M2, Prop.(a), p.342], $P\cdot\tilde{\Theta} = 2\Xi$, where Ξ is a natural model for the theta divisor of the Prym variety $P(\tilde{C}/C)$. We consider the pair (P,Ξ) to represent the Prym variety (P_0,ξ), just as we represent the Jacobian variety $(Pic^0(\tilde{C}),\tilde{\vartheta})$ by the pair $(Pic^{2g-2}(\tilde{C}), \tilde{\Theta})$. The restriction of the Abel map $\tilde{\alpha}:\tilde{C}^{(2g-2)}\to Pic^{2g-2}(\tilde{C})$ over P induces a surjection $\varphi:X\to\Xi$, where by definition $X = \tilde{\alpha}^{-1}(P) \subset \tilde{C}^{(2g-2)}$. By Riemann's singularities theorem, $\Xi\subset sing\tilde{\Theta}$, and in fact [M2, Prop.(b), p.342; SV1, lemma (0.3), p.343] the generic point of Ξ is a double point of $\tilde{\Theta}$, so by Abel's theorem the fiber of the Abel map $\tilde{\alpha}:\tilde{C}^{(2g-2)}\to Pic^{2g-2}(\tilde{C})$ over a generic point of Ξ is isomorphic to $\mathbb{P}^1$. In fact when C is non hyperelliptic, [Be3, Cor. of prop.3, p.365] X is a normal irreducible variety fibered via $\varphi:X\to\Xi$ over Ξ, such that if L is a smooth point of Ξ then $\varphi^{-1}(L) \cong \mathbb{P}^1$, and X is a $\mathbb{P}^1$ bundle in the Zariski topology over Ξ_{sm}. From the fact stated above that for all L on Ξ, $h^0(\tilde{C},L)$ is even, it follows that every fiber $\varphi^{-1}(L)$ is an odd dimensional projective space. It can be shown then that singular points L of Ξ occur then in two ways in relation to this fibration: either the fiber $\varphi^{-1}(L) \cong \mathbb{P}^r$ where $r \geq 3$, or L is the image of a singular point of X. Thus [SV6] $\mathrm{sing}\Xi = (\Xi \cap \mathrm{sing}_{\geq 4}\tilde{\Theta}) \cup \varphi(\mathrm{sing}X)$; i.e. L is singular on Ξ if and only if either $\dim.\varphi^{-1}(L) \geq 3$ or $\varphi^{-1}(L) \cap \mathrm{sing}X \neq \emptyset$. Moreover there is a "Riemann singularities theorem" for singularities not of the second type: i.e. we have $\mathrm{mult}_L\Xi = (r+1)/2$, whenever $\varphi^{-1}(L) \cap \mathrm{sing}X = \emptyset$ and $\dim.\varphi^{-1}(L) = r$. Denote $\mathrm{sing}_{st}\Xi = \{L: \dim.\varphi^{-1}(L) \geq 3\}$ = "stable singularities" of Ξ, and $\mathrm{sing}_{ex}\Xi = \{L: \varphi^{-1}(L) \cap \mathrm{sing}X \neq \emptyset\}$ = "exceptional singularities" of Ξ. Observe, as emphasized in [De1], that this decomposition of the singular locus of Ξ depends on the double cover, hence in the absence of a Torelli theorem is not necessarily intrinsic to Ξ.

The distinction between stable and exceptional singularities is crucial for application of the method of base loci of quadrics to Pryms, since Tjurin showed [T, p.957, line 11, lemma 2.3, p.963; cf. SV6, Prop.5.1] that the tangent cone at a stable double point of Ξ does contain the Prym canonical curve C_η, while at least in those cases where (P,Ξ) is a non hyperelliptic Jacobian the tangent cone at a general exceptional double point of Ξ does not contain C_η. It is entirely possible for the intersection $\mathrm{sing}_{st}\Xi \cap \mathrm{sing}_{ex}\Xi$ to be non empty, i.e. for a singularity to be both stable and exceptional. This just means for L on Ξ, that $\dim.\varphi^{-1}(L) \geq 3$ and also $\varphi^{-1}(L) \cap \mathrm{sing}X \neq \emptyset$. There is another important characterization of exceptional singularities due to Mumford [M2, case 1, bottom p.344], namely L on Ξ is an exceptional singularity if and only if L can be written as $\pi^*(M)(B)$ for some line bundle M on C with $h^0(C,M) \geq 2$, and some effective divisor $B \geq 0$ on $\tilde{C}$.

Remark: The presence of an exceptional singularity on Ξ has a simple geometric consequence for C. Since then $\omega_C \cong \mathrm{Nm}(L) \cong M^{\otimes 2}(\bar{B})$ where

$\overline{B} = Nm(B)$, it follows from [AM, Lemma 4, p.192] that a necessary condition for Ξ to have an exceptional singularity is that the canonical model C_ω lies on a rank 3 quadric. Moreover if C_ω lies on a rank 3 quadric and meets the vertex of that quadric, then [AM, Lemma 3, p.192] implies there is a line bundle M of degree < g-1 on C such that Mumford's conditions (a) and (b) [M2, top p. 345] hold, hence for any double cover of C, the theta divisor Ξ has an exceptional singularity. If C_ω lies outside the vertex on a rank 3 quadric, [AM, Lemma 3, p.192] implies there is an M on C such that $h^0(C,M) \geq 2$, and $M^{\otimes 2} \cong \omega_C$. Then $\pi^*(M) = L$ gives an exceptional singularity of Ξ if and only if Mumford's condition (c) [M2, top p. 345] also holds, i.e. that $h^0(C,M) + h^0(C,M\otimes\eta)$ is even.

3.2. Proposed Prym analogs of the three primary ingredients.

(i) "Enriques theorem": When is C_η the unique spanning curve in the base locus of the space $I_2(C_\eta)$ of all quadrics containing C_η? This is true for generic doubly covered curves C of genus $g \geq 8$ [De2] and for all doubly covered curves C with $g \geq 9$ and $Cliff(C) \geq 3$, [GL, LS]. Hence this part of the program for Donagi's conjecture is in place for $g \geq 9$. When $g = 8$ it remains open. When $g = 7$ a general curve C satisfies $Cliff(C) \geq 3$, but C_η lies on only 3 independent quadrics in $\mathbb{P}^5$, so this statement is <u>false</u> for $g = 7$, hence some other approach to Donagi's conjecture is necessary in that case.

(ii) "Riemann singularities package": When is it true that (a) $sing_{st}\Xi$ (is non empty and) has pure dimension p-6, (b) rank 6 stable double points are (open and) dense in $sing_{st}\Xi$, (c) the map

$\gamma: p \mapsto \mathbb{P}TC_p\Xi$ carries $sing_{2,st}\Xi$ into the space $\mathbb{P}(I_2(C_\eta))$, and (d) $sing_{st}\Xi$ can be recovered intrinsically from Ξ? If C is a general curve of genus $g \geq 7$, these are all true [We2, Ber, De2]. In fact $sing_{st}\Xi \neq \emptyset$ for all curves with $g \geq 7$ [Ber, Thm.1.4, p.670], and (c) holds for all non hyperelliptic C with $g \geq 4$, [T, p.963; cf. SV6 Prop. 5.1]. Assuming C is general, then $sing\Xi = sing_{st}\Xi$ in every genus [We2, Prop 3.5, p.682], $sing\Xi$ is irreducible of dimension p-6 if $g \geq 8$, and $sing\Xi$ is of (pure) dimension zero = p-6 if g = 7 [We2, Cor.3.7; De2, p.114, Thm. 1.1(i)]. This gives (a), (c) and (d) at least for general curves C with $g \geq 7$. For (b), if C is general with $g \geq 8$ then $sing\Xi = sing_{st}\Xi$ is irreducible, so it suffices to produce one rank 6 double point. The existence of double points L with $h^0(L) = 4$ follows from [We2, Lemma 3.2, p.681]. To show for general C these are all rank 6 double points one uses [We2, Prop. 3.5, p.682] as follows. We claim for general C the tangent cone to Ξ at L with $h^0(L) = 4$ is a rank 6 quadric. The square of an equation for this tangent cone is the determinant of the skew symmetric pairing $H^0(L)\times H^0(L) \to H^0(\omega_C(\eta)) \cong T_0{}^*(P)$, [M2, p.343; We2, p.682] provided that determinant is non zero. If $V = H^0(L)$, this pairing corresponds to a linear map $\beta:\Lambda^2(V) \to T_0{}^*(P)$, hence to the dual map $\beta^*:T_0P \to \Lambda^2(V^*)$. The determinant determines, up to scalar multiples, a degree 4 form on $\Lambda^2(V^*)$, which is the square of a rank 6 quadratic form Pf, the "universal Pfaffian", defined up to scalar multiples, which vanishes exactly on the locus in $\Lambda^2(V^*)$ corresponding to degenerate skew symmetric forms on V. Then the pullback $(\beta^*)^*(Pf) = q_L$ is an equation for the tangent cone to

Ξ at L, provided it is not identically zero, i.e. provided the image space $\beta^*(V)$ does not lie in the zero locus of Pf. By [We2, Thm. 1.11, p.674] for C general, β^* is surjective at every point L, so that q_L is not only non zero, but a rank six quadratic form on $T_0(P)$ whenever L is a double point. This proves that all double points have rank 6 for general C with $g \geq 7$, hence proves (b) for general C with $g \geq 8$. Moreover for general C of genus 7, although $\mathrm{sing}\Xi = \mathrm{sing}_{st}\Xi \neq \varnothing$, there are no triple points (or higher multiplicity) on Ξ by [We2, Cor. 1.10, Thm. 1.11, p.674; cf. SV6, Cor.3.3]. This proves (b) for general C with $g \geq 7$.

The precise analogs of (a), (b), and (d) are less complete.

<u>Assume C is any curve with $\mathrm{Cliff}(C) \geq 3$.</u> Then we have the following results: (a) If $g \geq 7$ and Z is any irreducible component of $\mathrm{sing}_{st}\Xi$, then $p-6 \leq \dim Z \leq p-5$, ([M2, Th., p.344; We2, Prop. 1.4, p.673, citing M1, and H, p.613]). No examples are known of curves with $\mathrm{Cliff}(C) \geq 3$, $g \geq 7$ and $\dim.\mathrm{sing}_{st}\Xi = p-5$, (and one might hope to show none exist). (b) If $g \geq 8$, double points are (open and) dense in $\mathrm{sing}_{st}\Xi$ (proved in Thm. 3.5 below). (d) If $g \geq 11$, $\mathrm{sing}_{st}\Xi$ is the union of those components of $\mathrm{sing}\Xi$ having dimension $\geq p-6$, [De1, Thm.3.1(i), p.547-8].

Thus the main questions remaining open, at least for $g \geq 11$ and $\mathrm{Cliff}(C) \geq 3$, are whether $\dim.\mathrm{sing}\Xi = p-6$, and whether "rank 6" double points are dense in $\mathrm{sing}_{st}\Xi$.

(iii) "Green's theorem": When does the image $\gamma(\mathrm{sing}_{2,st}\Xi)$ span $\mathbb{P}(I_2(C_\eta))$? This is true [De2, Thm.1.1.(ii), p.114] for generic $\tilde{C}/C$ with $g(C) \geq 7$. On the other hand, if $(J(W),\Theta(W))$ is the intermediate Jacobian of a cubic threefold W and $(J,\Theta) \cong (P(\tilde{C}/C),\Xi(\tilde{C}/C))$ is a general Prym representation with C a plane quintic [Be2; SV6, §5] (hence Cliff(C) = 1), then $I_2(C_\eta)$ contains the adjoints of an equation for W with respect to all points of some line on W, hence $I_2(C_\eta)$ has dimension ≥ 2, but singΞ is a single triple point, hence $\text{sing}_{2,st}(\Xi) = \varnothing$ is empty. There seem to be no known examples of double covers with Cliff(C) ≥ 3 for which $\gamma(\text{sing}_{2,st}\Xi)$ does not span $\mathbb{P}(I_2(C_\eta))$.

We propose the following.

3.3. Spanning conjectures.

(i) The set: $\gamma(\text{sing}_{2,st}\Xi) = \{Q_p|\ p \in \text{sing}_{2,st}\Xi\}$ spans $I_2(\tilde{C}/C)$, whenever double points are dense in $\text{sing}_{st}\Xi$.

(ii) If g(C) ≥ 8, and Cliff(C) ≥ 3, the set $\{p \in \text{sing}_{2,st}\Xi$ such that rank $Q_p = 6\}$ of "rank 6" double points, is dense in $\text{sing}_{2,st}\Xi$.

(iii) "rank 6 quadrics conjecture": If g(C) ≥ 8, and Cliff(C) ≥ 3, the set $\{Q_p|\ p \in \text{sing}_{2,st}\Xi$ and rank$Q_p = 6\}$ of "rank 6" quadrics spans $I_2(\tilde{C}/C)$.

Note that by Theorem 3.5 below, the hypothesis that double points are dense in $\text{sing}_{st}\Xi$ holds if g(C) ≥ 8 and dim.sing$\Xi(\tilde{C}/C) \leq p-5$, in particular when g(C) ≥ 8 and Cliff(C) ≥ 3, hence (i) and (ii) imply (iii). By Prop.3.6. below, the tangent cones at double points which are both stable and exceptional have rank ≤ 4. Hence in view of Debarre's results cited above in 3.2(ii), it would be safer to assume g ≥ 11 in 3.3(ii). At any rate it would be a significant advance to prove (i) or (ii) even assuming g ≥ 11 and $\text{dim.sing}_{st}\Xi = p-6$.

Remarks 3.4.

(i) By analogy with the argument for Jacobians, the generic "Enriques" result of Debarre in 3.2.(i) above already implies generic injectivity of the Prym map for $g \geq 8$. First one interprets the derivative of the Prym map in terms of the space $I_2(C_\eta)$ as follows [Be2, Prop.7.5, p.381]. If $\mathcal{P}:\mathcal{R}_g \to \mathcal{A}_{g-1}$ is the Prym map from the moduli space of doubly covered curves $\tilde{C}/C$ to the moduli space $\mathcal{A}_{g-1}$ of ppav's of dimension g-1, the subspace $\mathcal{P}_*(T_{\tilde{C}/C}\mathcal{R}_g) \subset T_{\mathcal{P}(\tilde{C}/C)}\mathcal{A}_{g-1}$ determines its ortho-complement $(\mathcal{P}_*(T_{\tilde{C}/C}\mathcal{R}_g))^\perp = \ker(\mathcal{P}^*)$ in $(T_{\mathcal{P}(\tilde{C}/C)}\mathcal{A}_{g-1})^*$. This inclusion in turn is equivalent to the inclusion $I_2(C_\eta) \subset S^2H^0(\omega_C \otimes \eta)$, hence determines a general C_η by Debarre's theorem. This proves $\mathcal{P}$ is generically injective for $g \geq 8$. The only generic Torelli argument valid for $g = 7$ seems to be the original one in [FS]. One lacks an argument for turning the precise analog of Enriques theorem in [GL, LS] into a precise Prym Torelli theorem. See however the next remark.

(ii) Here is a heuristic non constructive argument that Prym Torelli should be true whenever $g \geq 11$, $\mathrm{Cliff}(C) \geq 3$, and $\dim.\mathrm{sing}\Xi = p-6$. If $\mathcal{N}_{p-6} = \{(A,\Theta): \dim.\mathrm{sing}\Theta \geq p-6\} \subset \mathcal{A}_p$, the theorem on "normal generation" of Prym canonical line bundles [GL; LS, lemma 2.1, p.393], and generic Prym - Torelli, imply (by the interpretation in terms of the differential $\mathcal{P}_*$ of the Prym map in 3.4(i) above) that the Prym map $\mathcal{P}$ on such curves is a degree one immersion onto an open subset Z of a component of $\mathcal{N}_{p-6}$. Hence $\mathcal{P}$ can only fail to be injective over a singular point of Z. Moreover the precise "Enriques" theorem implies that even if two such double covers $\tilde{C}_1/C_1$ and $\tilde{C}_2/C_2$ define the same Prym variety (P,Ξ) on Z, the images $\mathcal{P}_*(T_{\tilde{C}_1/C_1}\mathcal{R}_g)$ and $\mathcal{P}_*(T_{\tilde{C}_2/C_2}\mathcal{R}_g)$

of the differential $\mathcal{P}_*$ are distinct in $T(P,\Xi)\mathcal{Q}_{g-1}$. I.e. the set Z of Prym varieties of curves with Cliff(C) ≥ 3 has distinct smooth branches at any possible singular point. For Donagi's conjecture, one wants to argue that all points of Z are in fact smooth points. It seems likely that if there were more than one branch of $Z\subset\mathcal{N}_{p-6}$ crossing itself at such a Prym variety, then the p-6 dimensional component of the cycle carried by the intrinsic singular scheme singΞ, would change under deformation along one of the branches of Z, for example a p-6 dimensional component of singΞ might disappear. If $g \geq 11$, Debarre's results in [De1, p.547-8] show all such components would be components of $\text{sing}_{st}\Xi$. On the contrary the calculations in [Ber; De2, Th.1.1, (i), p.114; DP; FP, chap.VIII] show the homology class of $\text{sing}_{st}\Xi$ is constant for all Prym varieties under consideration. Inferring a contradiction is however problematic, because these calculations are based on a "Brill Noether" type cycle structure for $\text{sing}_{st}\Xi$ which assumes a choice of double cover $\tilde{C}/C$ representing (P,Ξ) as a Prym variety. The homology class which would accurately reflect the behavior of singΞ under deformation would seem to be one based on the intrinsic scheme structure of singΞ. (Compare with Remark 2.3.(ii) above.)

Density of stable double points in $\text{sing}_{st}\Xi(\tilde{C}/C)$

Theorem 3.5 below is an improvement of the density theorem for double points on the stable, non exceptional, singular locus for Prym theta divisors proved in [SV6]. A primary tool used in that paper was to prove that "RST" holds at any non exceptional singularity of Ξ, i.e. if

L is on Ξ but not an exceptional singularity, then $\mathrm{mult}_L\Xi$ = $(1/2)h^0(\tilde{C},L)$. The proof of that theorem is as follows. As a set Ξ = $\tilde{\Theta}\cap P$, and the RST holds at L on Ξ if and only if the projectivized tangent cone $\mathbb{P}TC_L\Xi$ to Ξ at L is set theoretically the intersection $\mathbb{P}(T_LP\cap TC_L\tilde{\Theta})$ of $\mathbb{P}T_LP$ with the projectivized tangent cone to $\tilde{\Theta}$ at L. Using the Abel parametrization $\varphi:X\to\Xi$ one shows first that the projectivized intersection $\mathbb{P}(T_LP\cap TC_L\tilde{\Theta})$ is the projectivized image under the differential φ_* of the union of the Zariski normal <u>spaces</u> to X at points of the fiber $\varphi^{-1}(L)$. Since the tangent cone $\mathbb{P}TC_L\Xi$ is always the image of the projectivized normal <u>cone</u> to $\varphi^{-1}(L)$ in X, it follows that RST holds at L if the projectivized normal cone to $\varphi^{-1}(L)$ in X equals the union of the Zariski normal spaces to X at all points of $\varphi^{-1}(L)$. This is true if X is smooth along $\varphi^{-1}(L)$, i.e. if L is not an exceptional singularity. This permits us to compute the multiplicity of a non exceptional point L on Ξ from the dimension of $\varphi^{-1}(L)$. Using this, and an argument that at a generic point L of a component of the stable singular locus the fiber is $\varphi^{-1}(L) \cong \mathbb{P}^3$ [SV6, Cor. 3.3], it follows that double points are dense in the set of stable, non exceptional, singularities of Ξ when $g(C) \geq 6$ and $\dim.\mathrm{sing}\Xi \leq p-5$. We strengthen this density result below to hold for the set of all stable singularities of Ξ, at least for $g(C) \geq 8$ and $\dim.\mathrm{sing}\Xi \leq p-5$. It is hopeless to prove this statement for $g(C) = 6$ since double points are in fact not dense on the stable singular locus of the theta divisor of the intermediate Jacobian of a cubic threefold, realized as the Prym variety of a doubly covered smooth plane quintic. The key ingredient is a new Riemann singularities theorem, Prop. 3.6 below, for certain singularities which are both stable and exceptional on a Prym theta divisor.

3.5. Theorem. Let C be a smooth curve of genus $g \geq 8$ and assume that $\dim.\mathrm{sing}\Xi \leq p-5$, i.e. that C is not trigonal, not hyperelliptic, and not a double cover of an elliptic curve. If $\tilde{C} \to C$ is any connected double cover and (P,Ξ) the corresponding Prym variety, then $\mathrm{sing}_{2,st}(\Xi)$ is dense in $\mathrm{sing}_{st}(\Xi)$. [Recall that the set $\mathrm{sing}_{st}(\Xi)$ may depend on the choice of the étale double cover $\tilde{C}/C$ realizing (P,Ξ).]

Proof: First, recall that every irreducible component of $\mathrm{sing}_{st}(\Xi)$ has dimension $\geq p-6 = g - 7$. Then by [SV6, Cor. 3.3], for a general point L of any irreducible component of $\mathrm{sing}_{st}(\Xi)$, we have $h^0(\tilde{C},L) = 4$. If $h^0(L) = 4$ and L is not an "exceptional singularity" of $\Xi(\tilde{C}/C)$, then L is a double point of Ξ by [SV6, Thm. 2.1], and hence stable double points are dense in any irreducible component of $\mathrm{sing}_{st}(\Xi)$ that is not also contained in $\mathrm{sing}_{ex}(\Xi)$. Thus it only remains to worry about irreducible components of $\mathrm{sing}_{st}(\Xi)$ that also consist entirely of exceptional singularities. We will use Mumford's characterization of exceptional singularities [M2, p.344]. I.e. L on Ξ is an exceptional singularity if and only if L can be written as $\pi^*(M)(B)$ for some line bundle M on C with $h^0(C,M) \geq 2$, and some divisor $B \geq 0$ on $\tilde{C}$.

Next, denote as usual by $W^r{}_d$ the set of line bundles M on C with degree d and $h^0(C,M) \geq r+1$, and recall from Martens-Mumford [ACGH, Thms. 5.1,5.2, pp. 191-193] that with the hypotheses of the theorem and $4 \leq d < g-1$, we have $\dim(W^2{}_d(C)) < d-5$. Then, as in [M2, top p. 347], one gets the estimate:

$\dim \{(M,B) \mid M \in \mathrm{Pic}^d(C),\ 4 \leq d < g-1,\ h^0(M) \geq 3,\ \mathrm{Nm}(B) \in |\omega_C - 2M|\} \leq$

$\dim(W^2{}_d(C)) + \dim(|\omega_C - 2M|) < (d-5) + (1/2)\deg(\omega_C - 2M) - 1 =$ $(d-5) + (1/2)(2g-2-2d) - 1 = d-5 + g-1-d - 1 = g-7 = p-6$. Since every irreducible component of $\mathrm{sing}_{st}(\Xi)$ has dimension $\geq p-6 \geq 1$, and we have argued that the set of exceptional singularities of form $\pi^*(M)(B)$ with M in $W^2{}_d(C)$, $4 \leq d < g-1$ have dimension $< p-6$, it follows that such exceptional singularities are not dense in any component of $\mathrm{sing}_{st}(\Xi)$. Also the case $d = g-1$ can only give a finite number of exceptional singularities, since then $M^2 \cong \omega_C$, hence for $g \geq 8$ and thus $p-6 \geq 1$, these cannot be dense in any component of $\mathrm{sing}_{st}\Xi$. By hypothesis, $W^1{}_d(C)$ is empty for $d \leq 3$ since C is not trigonal. Thus it only remains to show double points are dense in any irreducible component of $\mathrm{sing}_{st}(\Xi)$ such that the general point L has $h^0(L) = 4$ and is exceptional but cannot be written in the form $\pi^*(M)(B)$ for $h^0(M) \geq 3$. Proposition 3.6 below thus will complete the argument for the density theorem.

Terminology: We say that "Riemann singularities theorem (RST) holds at L (on $\Xi(\tilde{C}/C)$)" if and only if $\mathrm{mult}_L\Xi = (1/2)h^0(\tilde{C},L)$. Notice that this statement is relative to the representation of $(P,\Xi) \cong (P(\tilde{C}/C), \Xi(\tilde{C}/C))$ by the particular double cover $\pi:\tilde{C}\to C$. Thus one double cover representing (P,Ξ) may compute the multiplicity of Ξ at L and another one may not.

3.6 Proposition: Let $\pi: \tilde{C} \to C$ be a connected étale double cover of a smooth curve of genus $g \geq 3$. Let $M \in \mathrm{Pic}^d(C)$ for any $2 \leq d \leq g-1$ have $h^0(M) = 2$ and let B be a divisor ≥ 0 on $\tilde{C}$ such that B and $\iota^*(B)$ have disjoint supports and $L = \pi^*(M)(B)$ satisfies $\mathrm{Nm}(L) = \omega_C$ and $h^0(L) = 4$.

Then L is a double point of Ξ and has rank ≤ 4, i.e. the RST holds at L and the Pfaffian quadratic form defining the tangent cone $TC_L(\Xi)$ has rank at most 4.

3.7 Remarks: **(i)** Note that if the divisor B on $\tilde{C}$ were allowed to have the more general form $\pi^*(B_0)+B_1$, where $B_0 \geq 0$ on C with $B_1 \geq 0$ having the property that B_1 and $\iota^*(B_1)$ have disjoint supports, one could absorb B_0 into M, i.e. rewrite $L = \pi^*(M)(B)$ as $\pi^*(M(B_0))(B_1)$. Then there are two possibilities: if $h^0(M(B_0)) = 2$, then the Proposition applies to $L = \pi^*(M(B_0))(B_1)$ and proves L is a double point on Ξ. If $h^0(M(B_0)) > 2$, then the hypotheses of the Proposition do not apply, and indeed the multiplicity of Ξ at L is at least 3 by Shokurov's argument [Sh, Lemma 5.7, p. 121]. In this case however, since L can be written as $\pi^*(M_0)(B_1)$, with $M_0 = M(B_0)$ and $h^0(M_0) \geq 3$, such points cannot be dense in $\mathrm{sing}_{st}(\Xi)$ under the hypotheses of the theorem. Thus, in conjunction with Shokurov's argument, this proposition shows when $h^0(\tilde{C},L) = 4$, that RST holds at L if and only if L cannot be written as $\pi^*(M)(B)$ where $h^0(M) > 2$, $B \geq 0$. In other words L on Ξ is a stable double point if and only if $h^0(\tilde{C},L) = 4$ and L cannot be written as $\pi^*(M)(B)$ where $h^0(M) > 2$, $B \geq 0$.

(ii) The proof of Lemma 3.8 below is based on the argument in [M2, p. 343]. We recall next Mumford 's notation for the restriction to T_LP of Kempf's equation for $TC_L\tilde{\Theta}$. By calculating directly with the restricted equation, Mumford [M2, Prop., p.343] showed (assuming $NmL = \omega_C$) if $h^0(L) = 2$ the restriction is zero precisely when L has form $\pi^*(M)(B)$ where $h^0(M) = 2$, $B \geq 0$. We will show if $h^0(L) = 4$, the restriction of Kempf's equation to T_LP is zero, i.e. RST fails at L, precisely when L can be written in the form form $L = \pi^*(M)(B)$ where $h^0(M) \geq 3$, $B \geq 0$.

(iii) As remarked in [SV6, Rmk 2.3(i)], Shokurov's argument immediately generalizes to show that RST fails at L on Ξ if L can be written in the form $L = \pi^*(M)(B)$ where $h^0(M) > (1/2)h^0(L)$, $B \geq 0$. It seems plausible that the converse is also true, i.e. that RST holds at L on Ξ provided L cannot be written in the form $\pi^*(M)(B)$ where $h^0(M) > (1/2)h^0(L)$, $B \geq 0$. We can prove this in the extremal case when L <u>can</u> be written as $\pi^*(M)(B)$, where $h^0(M) = (1/2)h^0(L)$, $B \geq 0$.

Notation:

(1) For line bundles L and M and sections $s \in H^0(L)$ and $t \in H^0(M)$, we use the notation $s \cdot t$ for the cup product in $H^0(L \otimes M)$. (Notice that $s \cdot t = t \cdot s$ under the identification $H^0(L \otimes M) \cong H^0(M \otimes L)$ induced by the standard isomorphism $L \otimes M \cong M \otimes L$ of line bundles.)

(2) For $L \in \Xi$ and $(s,t) \in H^0(L) \times H^0(L)$, let $\langle s,t \rangle = s \cdot \iota^*(t) \in H^0(\tilde{\omega})$. (Namely, following [M2, p. 343], since $Nm(L) \cong \omega_C$, then $\iota^*(L) \cong \tilde{\omega} \otimes L^*$ (since $L \otimes \iota^*(L) \cong \pi^*(Nm(L)) \cong \tilde{\omega}$) so $H^0(L) \cong H^0(\iota^*(L)) \cong H^0(\tilde{\omega} \otimes L^*)$ and the standard cup product pairing $H^0(L) \times H^0(\tilde{\omega} \otimes L^*) \to H^0(\tilde{\omega})$ yields the pairing $H^0(L) \times H^0(L) \to H^0(\tilde{\omega})$, $(s,t) \mapsto \langle s,t \rangle$. Explicitly, $H^0(L) \times H^0(L) \cong H^0(L) \times H^0(\iota^*(L)) \to H^0(L \otimes \iota^*(L)) \cong H^0(\tilde{\omega})$, equivalently, $H^0(L) \times H^0(L) \cong H^0(L) \times H^0(\iota^*(L)) \cong H^0(L) \otimes H^0(\tilde{\omega} \otimes L^*) \to H^0(\tilde{\omega})$.)

(3) Then we let $\beta(s,t)$ denote $(\langle s,t \rangle - \langle t,s \rangle)$, so the map $\beta: H^0(L) \times H^0(L) \to H^0(\omega \otimes \eta)$ is skew-symmetric. [Here $H^0(\omega \otimes \eta) \subset H^0(\tilde{\omega})$ is the (-1) eigenspace for the action of ι^* on $H^0(\tilde{\omega})$; see [M2, p. 343; We2 p. 673]. Welters [We2, (1.8)] uses β to denote (1/2) the corresponding linear map $\Lambda^2(H^0(L)) \to H^0(\omega \otimes \eta)$ and calls it the "Prym-

Petri map for L".]

3.8 Lemma: Assume L in Ξ has form $\pi^*(M)(B)$, $B \geq 0$, and that B and $\iota^*(B)$ have disjoint supports; let u denote a section of $\mathcal{O}(B)$ vanishing precisely on B, and in $H^0(\pi^*(M)(B))$ consider any $\sigma \in u\cdot\pi^*(H^0(M))$, $\sigma \neq 0$, and any $\tau \notin u\cdot\pi^*(H^0(M))$. Then $\beta(\sigma,\tau) \neq 0$ in $H^0(\omega\otimes\eta)$.

Proof: First write $\sigma = \pi^*(s_0)\cdot u$ for $s_0 \in H^0(M)$, $s_0 \neq 0$. If $\beta(\sigma,\tau) = 0$, then $\langle\sigma,\tau\rangle = \langle\tau,\sigma\rangle$, i.e. $(\pi^*(s_0)\cdot u)\cdot\iota^*(\tau) = \tau\cdot(\pi^*(s_0)\cdot\iota^*(u))$ in $H^0(\tilde{\omega})$ (since $\iota^*(\pi^*(s_0)) = \pi^*(s_0)$). Then, canceling the (nonzero) factor $\pi^*(s_0)$ from both sides, we have $u\cdot\iota^*(\tau) = \tau\cdot\iota^*(u)$ in $H^0(\tilde{\omega}\otimes(\pi^*(M))^*) \cong H^0(\pi^*(M)(B+\iota^*(B)))$, since $\tilde{\omega} = \pi^*(M)(B)\otimes\iota^*(\pi^*(M)(B)) = \pi^*(M)(B)\otimes\pi^*(M)(\iota^*(B))$. Since the divisor of u is B, the equation $u\cdot\iota^*(\tau) = \tau\cdot\iota^*(u)$ implies the equation $B + (\iota^*(\tau)) = (\tau) + \iota^*(B)$ on divisors. Then, since $(\iota^*(\tau))$ is effective, we have $B \leq B + (\iota^*(\tau)) = (\tau) + \iota^*(B)$; thus $B \leq (\tau) + \iota^*(B)$, but since B and $\iota^*(B)$ have disjoint supports, we must have $B \leq (\tau)$. Therefore, since τ is a regular section of $\pi^*(M)(B)$, u is a regular section of $\mathcal{O}(B)$, and $(u) \leq (\tau)$, thus τ/u is a regular section of the line bundle $\pi^*(M)$ on $\tilde{C}$. Now $H^0(\tilde{C},\pi^*(M)) \cong H^0(C,M)\oplus H^0(C,M(\eta))$ and ι^* acts by id on $H^0(C,M)$ and by -id on $H^0(C,M(\eta))$. But $\iota^*(\tau/u) = \tau/u$ (from the above identity $u\cdot\iota^*(\tau) = \tau\cdot\iota^*(u)$), so τ/u lies in the $H^0(C,M)$ summand, i.e. $\tau/u = \pi^*(t_0)$ for $t_0 \in H^0(M)$. Then $\tau = u\cdot\pi^*(t_0) \in u\cdot\pi^*(H^0(M))$, contradiction. **QED** for Lemma 3.8.

Now we complete the proof of Proposition 3.6. So assume as in the proposition that $L = \pi^*(M)(B)$, $h^0(L) = 4$ and $h^0(M) = 2$, and B and

$\iota^*(B)$ have disjoint supports. Look at the Pfaffian obtained as a square root of the restriction of an equation for the tangent cone of $\tilde{\Theta}$ at L, to T_LP. Since $h^0(L) = 4$, if the Pfaffian is not identically zero on T_LP, then it defines a degree two equation for the tangent cone of Ξ at L.

Recall [M2, pp.342-3] how to get this Pfaffian: take a basis for $H^0(L)$ and a basis for sections of $H^0(\tilde{\omega}\otimes L^*)$, tensor them together to obtain elements in $H^0(\tilde{\omega})$, and restrict these down from $T_L\tilde{J}$ to T_LP. Recall the bracket $\langle s,\alpha\rangle = s\cdot\iota^*\alpha$ for s,α in $H^0(L)$, where $\iota^*\alpha$ is in $H^0(\tilde{\omega}\otimes L^*)$, i.e. $\iota^*L \cong \tilde{\omega}\otimes L^*$ because of the assumption $Nm(L) = \omega$, and $\langle s,\alpha\rangle$ is in $H^0(\tilde{\omega})$. Then choose a basis s_1,s_2 of $H^0(C,M)$, let u be a section cutting out the divisor B, and denote $\sigma_1 = u\cdot\pi^*(s_1)$, $\sigma_2 = u\cdot\pi^*(s_2)$, the two corresponding elements of $H^0(L)$. Complete this to a basis of $H^0(L)$ by elements τ_1,τ_2, which therefore do not belong to $u\cdot\pi^*(H^0(M))$. The Pfaffian is a square root of the determinant of a skew symmetric 4×4 matrix whose upper left hand 2×2 block has entries $\beta(\sigma_i,\sigma_j)$, $1\le i,j \le 2$, and whose upper right hand 2×2 block has entries $\beta(\sigma_i,\tau_j)$, $1\le i,j \le 2$. We claim the upper left 2×2 block is zero because the pulled back sections $\pi^*(s_i)$ are invariant under the involution ι of $\tilde{C}$. Eg. the (1,2) entry of this block is $\beta(\sigma_1,\sigma_2) = \langle\sigma_1,\sigma_2\rangle - \langle\sigma_2,\sigma_1\rangle = \sigma_1\cdot\iota^*\sigma_2 - \sigma_2\cdot\iota^*\sigma_1 = u\pi^*s_1\cdot\iota^*(u\cdot\pi^*s_2) - u\pi^*s_2\cdot\iota^*(u\pi^*s_1) = u\pi^*s_1\cdot\iota^*(u)\pi^*s_2 - u\pi^*s_2\cdot\iota^*(u)\pi^*s_1$. This equals zero as a section of the line bundle $\tilde{\omega}$, (compute it locally where these products become multiplication in $\mathbb{C}$, hence commute). Then the Pfaffian equals the determinant of the upper right hand 2×2 block. Hence it suffices to check this determinant is non zero.

From the description above of the upper right block, this determinant has form AD-BC where the entries A = $\beta(\sigma_1,\tau_1)$, B = $\beta(\sigma_1,\tau_2)$, C = $\beta(\sigma_2,\tau_1)$, D = $\beta(\sigma_2,\tau_2)$ are all viewed as linear forms on T_LP = $H^0(\omega\otimes\eta)^*$. First, all four linear forms A,B,C,D are non zero by the lemma. Then assume that AD-BC = 0 so that AD = BC. Then since A is an irreducible element of the symmetric algebra on $H^0(\omega\otimes\eta)$, thought of as a polynomial ring with $H^0(\omega\otimes\eta)$ as its linear forms, A divides either B or C, so either A = λB or A = λC, where λ is a non zero scalar. If A = λC, take σ_1, and $\sigma_1-\lambda\sigma_2$ as new basis for $u\pi^*H^0(M)$. Then $\beta(\sigma_1-\lambda\sigma_2,\tau_1) = \beta(\sigma_1,\tau_1) - \lambda\beta(\sigma_2,\tau_1)$ = A-λC = 0. This contradicts the lemma applied to the matrix formed from this new basis. If A = λB, then take τ_1, $\tau_1-\lambda\tau_2$ as new basis for the space spanned by τ_1,τ_2, complementary to $u\pi^*H^0(M)$. Then the upper right hand block of the new matrix has as (1,2) entry $\beta(\sigma_1,\tau_1-\lambda\tau_2) = \beta(\sigma_1,\tau_1) - \lambda\beta(\sigma_1,\tau_2)$ = A-λB = 0, a contradiction. This proves Proposition 3.6. **QED.**

3.9. Conjecture. If $\tilde{C}/C$ is a double cover with Prym variety (P,Ξ), and L is a point of Ξ which cannot be written as $\pi^*(M)(B)$ where $h^0(C,M) > (1/2)h^0(\tilde{C},L)$, $B \geq 0$, then RST holds at L, i.e. $\mathrm{mult}_L\Xi = (1/2)h^0(\tilde{C},L)$.

A deformation theoretic formulation of the spanning condition

A key idea in the paper [AM] is that a necessary condition for a double point x of a theta divisor Ξ to persist under a given first order deformation, is that the direction of the deformation must lie within the hyperplane tangent to moduli $\mathcal{Q}_p$ which is defined by the quadric tangent cone at the double point x, [cf. M3, p. 87; SV3, pp.241-2]. The resulting link between the spanning problem for Jacobians and deformation theory was made formally precise in [SV2] and used there to give a proof of Green's theorem as sketched above. In the interest of ultimately proving the spanning conjecture for Prym varieties, we give here a corresponding precise version of this link for Prym varieties which grew out of discussions with O. Debarre from 1990. Let $\tilde{C}/C$ denote an étale double cover $\pi: \tilde{C} \to C$ of connected, smooth curves, where $g(C) = g$. Let $P = P(\tilde{C}/C)$ denote the Prym variety of $\tilde{C}/C$ and $p = \dim(P) = g-1$. Let $\mathcal{P}: \mathcal{R}_g \to \mathcal{A}_p$ be the "Prym map" $\tilde{C}/C \mapsto (P,\Xi)$, and $\mathcal{P}_p = \mathcal{P}(\mathcal{R}_g) \subset \mathcal{A}_p$ be the (classical) Prym locus. Let $T_{\tilde{C}/C}(\mathcal{P}_p)$ denote the image $\mathrm{im}(\mathcal{P}_*) \subset T_P(\mathcal{A}_p)$ of the differential on the level of moduli functors, which means we identify the differential $\mathcal{P}_*$ of the Prym map, with the operation of inducing a first order deformation of P from a first order deformation of $\tilde{C}/C$, (cf. [Be2, p.381]). If $I_2(C_\eta)$ denotes the vector space of quadratic polynomials vanishing on the Prym canonical curve $C_\eta \subset \mathbb{P}^{p-1}$, and if $x \in \mathrm{sing}_{st,2}(\Xi)$ = {stable double points on Ξ}, recall the quadric tangent cone Q_x to Ξ at x belongs to $\mathbb{P}I_2(C_\eta)$ [T, cf. SV6]. Next we recall the relevant tools from deformation theory [SV2]. The object that controls the deformations of Ξ is $\mathcal{T}^1(\Xi)$ = the sheaf of 1st order deformations of Ξ. $\mathcal{T}^1$ is quite general, defined for any scheme, but in the case of a global hypersurface such as $\Xi \subset P$, $\mathcal{T}^1(\Xi)$ is a line bundle, supported on $\mathrm{sing}(\Xi)$, the scheme defined by the functions ξ and $\partial\xi/\partial z_i$, $i = 1,\ldots,p$, and there is a corresponding global presentation of it in terms of more familiar sheaves as follows:

$$0 \to \mathcal{T}(\Xi) \to \mathcal{T}(P)|_\Xi \to \mathcal{N}(\Xi/P) \to \mathcal{T}^1(\Xi) \to 0 .$$

That is, for a hypersurface in a smooth ambient space, $\mathcal{T}^1$ is the sheaf cokernel of the map from the restricted tangent bundle of the ambient to the normal sheaf of the hypersurface. For Ξ, the bundle $\mathcal{T}(P)|_{\Xi}$ is trivial, $\mathcal{N}(\Xi/P)$ is $\mathcal{O}_{\Xi}(\Xi)$, and the map between them takes $\partial/\partial z_i$ to $\partial\xi/\partial z_i$, and so $\mathcal{T}^1(\Xi)$ is just the restriction of $\mathcal{O}(\Xi)$ to the singular locus:

$$\mathcal{O}_{\Xi}(\Xi)|_{\mathrm{sing}(\Xi)} = \mathcal{T}^1(\Xi).$$

Locally of course any line bundle on $\mathrm{sing}(\Xi)$ is isomorphic to the structure sheaf of $\mathrm{sing}(\Xi)$, and this isomorphism can be made explicit here as follows:

$f\in\mathcal{O}_P/(\xi,\partial\xi/\partial z_1,\ldots,\partial\xi/\partial z_p) \longleftrightarrow$ the 1^{st} order deformation $\{\xi(z)+\varepsilon f(z)=0\}$.

For example at a rank 6 stable double point, locally analytically $\xi \approx z_1{}^2+\ldots+z_6{}^2$, and the first order deformation given by f is: $z_1{}^2+\ldots+z_6{}^2+\varepsilon f(z_7,\ldots,z_p)$.

A fundamental fact is the existence of a universal family of theta divisors, i.e. a universal global deformation of Ξ:

$$\begin{array}{ccc} \Xi & \subset & \boldsymbol{\Xi} \\ \downarrow & & \downarrow \\ P & \in & \mathcal{H}_p \end{array}$$

parametrized by the Siegel "upper half-space": $\mathcal{H}_p$ = {p×p symmetric complex matrices Ω s.t. $\mathrm{Im}\Omega > 0$}. $\boldsymbol{\Xi}$ here is the hypersurface defined by the equation $\xi = 0$ with ξ considered as a function of both variables (z,Ω). This implies that we have a Kodaira-Spencer map:

$$\begin{aligned} s\colon\ T_P(\mathcal{H}_p) &\to H^0(\mathcal{T}^1(\Xi)) \\ \partial/\partial\Omega_{ij} &\mapsto \partial\xi/\partial\Omega_{ij} \end{aligned}$$

(i.e. since we have an actual family of global deformations, to each direction in $\mathcal{H}_p$ we get a collection of 1^{st} order local deformations of Ξ that fit together, and may be represented by the directional derivative of the global equation for Ξ). The image of this map therefore defines a linear system on sing(Ξ), and consequently a rational map from sing(Ξ) to the projective space $\mathbb{P}^* H^0(\mathcal{T}^1(\Xi))$.

Now the analysis of sing(Ξ) and $H^0(\mathcal{T}^1(\Xi))$ is complicated by the possibly non-reduced structure at higher multiplicity points and double points of rank < 6. Furthermore we want to use deformation theory to reformulate the problem of determining the span of the set of tangent cones at stable double points of Ξ. So to simplify things, let $\mathrm{sing}_{st}(\Xi) \subset \mathrm{sing}\Xi$ denote the reduced closed subscheme of $\mathrm{sing}\Xi$ consisting of stable singular points of the Prym theta divisor $\Xi = \Xi(\tilde{C}/C)$ and put

$$\mathcal{T}^1{}_{st}(\Xi) = \text{the restriction of } \mathcal{T}^1(\Xi) \text{ to } \mathrm{sing}_{st}(\Xi).$$

Then sections of $\mathcal{T}^1{}_{st}(\Xi)$ can be described by giving their values at stable singular points. It remains to be seen of course that in the problem at hand this simplification does not throw away too much information. Recall stable double points correspond to certain complete linear series $g^3{}_{2p}$ of divisors on the curve $\tilde{C}$ whose images on C are canonical divisors.

Proposition 3.10. Let $P = P(\tilde{C}/C) \in \mathcal{P}_p$ be the Prym variety of $\tilde{C}/C$, let $\bar{s}$: $T_P\mathcal{G}_p \to H^0(\mathcal{T}^1{}_{st}(\Xi))$ be the reduced Kodaira-Spencer map defined above by the composition $T_P(\mathcal{H}_p) \to H^0(\mathcal{T}^1(\Xi)) \to H^0(\mathcal{T}^1{}_{st}(\Xi))$. Then,

(i) $T_{\tilde{C}/C}(\mathcal{P}_p) \subset \ker(\bar{s})$.

(ii) The set $\{Q_x | \, x \in \mathrm{sing}_{st,2}(\Xi)\}$ spans $\mathbb{P}I_2(C_\eta) \Leftrightarrow [T_{\tilde{C}/C}(\mathcal{P}_p) = \ker(\bar{s})]$.

(iii) In fact, the following quotient vector spaces are dual:

$$\ker(\bar{s})/T_{\tilde{C}/C}(\mathcal{P}_p) \text{ and } I_2(C_\eta)/\mathrm{span}\{q_x | \, x \in \mathrm{sing}_{st,2}(\Xi)\},$$

where q_x is an equation for Q_x.

Proof of Prop.3.10. (i): If $v = \Sigma a_{ij} \partial/\partial\Omega_{ij}$ is a direction in $T_P\mathcal{Q}_p$, then $\overline{s}(v)$ is the section of $\mathcal{T}^1{}_{st}(\Xi)$ whose value at x in $sing_{st}\Xi$ is $\Sigma a_{ij} \partial^2\xi/\partial z_i\partial z_j$ (x) where ξ is a theta function defining Ξ and z_i are variables on the universal cover of P and the derivatives are evaluated at x. The heat equation is used here to replace the section $\overline{s}(v) = \Sigma a_{ij} \partial\xi/\partial\Omega_{ij}$ of $\mathcal{T}^1{}_{st}(\Xi)$ by the section $\Sigma a_{ij} \partial^2\xi/\partial z_i\partial z_j$ of $\mathcal{O}_\Xi(\Xi)$ on $sing_{st,2}(\Xi)$.

Take $v \in T_{\tilde{C}/C}(\mathcal{P}_p) = im(\mathcal{P}_*)$ and consider $\overline{s}(v) \in H^0(\mathcal{T}^1{}_{st}(\Xi))$. We claim $\overline{s}(v) = 0$. If x is a stable singular point at which Ξ has multiplicity ≥ 3, then $\overline{s}(v)(x) = 0$ because we are evaluating a linear function on the quadratic part of the theta function at x, which is zero in this case. Thus it is equivalent to check that $\overline{s}(v)(x) = 0$ for all $x \in sing_{st,2}(\Xi)$, i.e. that $v(q_x) = 0$ for all $x \in sing_{st,2}(\Xi)$. Here, we regard v as a linear functional on the space of quadratic forms on $\mathbb{P}^{p-1} = \mathbb{P}T_0(P)$ via the natural identification

$$\begin{array}{c} \mathbb{P}T^*{}_P\mathcal{Q}_p = \text{projectivized cotangent space to } \mathcal{Q}_p \text{ at } P \\ \wr\| \\ \mathbb{P}S^2T^*{}_0P = \{\text{quadrics in } \mathbb{P}^{p-1}\} \end{array}$$

which induces an isomorphism $\mathbb{P}(T_{\tilde{C}/C}(\mathcal{P}_p))^\perp = \mathbb{P}\ker(\mathcal{P}^*) \cong \mathbb{P}I_2(C_\eta)$:

$$\begin{array}{ccccc} & & \mathbb{P}T^*{}_P\mathcal{Q}_p & \supset & \mathbb{P}(T_{\tilde{C}/C}(\mathcal{P}_p))^\perp \\ & & \wr\| & & \wr\| \\ \{\text{quadrics in } \mathbb{P}^{p-1}\} & = & \mathbb{P}S^2T^*{}_0P & \supset & \mathbb{P}I_2(C_\eta) = \{\text{quadrics containing } C_\eta\} \end{array}$$

Since by the diagram a quadric Q in $\mathbb{P}^{p-1}$ is orthogonal to $\mathbb{P}(T_{\tilde{C}/C}(\mathcal{P}_p))$ if and only if Q contains C_η, to see that $v(q_x) = 0$ for all $x \in sing_{st,2}(\Xi)$

when v is in $\mathbb{P}(T_{\tilde{C}/C}(\mathbb{P}_p))$, it suffices to know that $Q_x \supset C_\eta$ for all $x \in \mathrm{sing}_{st,2}(\Xi)$. This is the result of Tjurin recalled above [T, cf. SV6].
QED (i).

Proof of Prop.3.10. (ii) and (iii): Consider the following inclusions of subspaces:
$T_{\tilde{C}/C}(\mathbb{P}_p) \subset \ker(\bar{s}) \subset T_p\mathcal{G}_p$, and
$\mathrm{span}\{q_x | x \in \mathrm{sing}_{st,2}(\Xi)\} \subset I_2(C_\eta) \subset T^*_p\mathcal{G}_p$,
where q_x is an equation for Q_x. The arguments above show that:
(a) $\{q_x | x \in \mathrm{sing}_{st,2}(\Xi)\}^\perp = \ker(\bar{s})$,
and the isomorphism from [Be2, p.381] quoted above proves that
(b) $I_2(C_\eta) = (T_{\tilde{C}/C}(\mathbb{P}_p))^\perp$. Hence the natural pairing on the spaces $T_p\mathcal{G}_p$ and $T^*_p\mathcal{G}_p$ induces a non degenerate pairing between $\ker(\bar{s})/(T_{\tilde{C}/C}(\mathbb{P}_p))$ and $I_2(C_\eta)/\mathrm{span}\{q_x\}$. **QED.**

Remarks 3.11. (i) This gives a precise version of the informal statement that the spanning condition holds if and only if the only deformation directions in which all stable double points persist as singularities on Ξ, are the directions induced by deformations of the double cover $\tilde{C}/C$.
(ii) This equivalence is illustrated by the case of the intermediate Jacobian of a cubic threefold realized as the Prym of a doubly covered plane quintic C, since both sides of the statement in (2) fail. Indeed the set $\{Q_x | x \in \mathrm{sing}_{st,2}(\Xi)\}$ is empty while $\mathbb{P}I_2(C_\eta)$ contains the $\mathbb{P}^1$ of quadrics adjoint to the cubic threefold W along a general line in W. The fiber of the Prym map at $\tilde{C}/C$ contains the Fano surface of W

hence $T_{\tilde{C}/C}(\mathcal{P}_p)$ has codimension ≥ 2 in $\ker(\bar{s}) = T_p\mathcal{G}_p$.

§4. Andreotti's approach to Torelli via the Gauss map on the theta divisor

Recall the usual formulation of Andreotti's argument for (non hyperelliptic) Jacobians [M3,p.79; ACGH, p.246]: the (rational) Gauss map $\gamma:\Theta\text{---->}|\omega_C| \cong \mathbb{P}^{g-1*}$ on the theta divisor, sends a smooth point L of Θ to the projectivized tangent space $\mathbb{P}T_L\Theta$, to Θ at L, translated to the origin of $Pic^0(C)$, which thus defines a hyperplane $\mathbb{P}T_L\Theta \subset |\omega_C|^* \cong \mathbb{P}^{g-1}$ in the ambient space for the canonical model $C_\omega \subset \mathbb{P}^{g-1}$ of the curve C. Parametrizing the theta divisor by the birational Abel map $\alpha:C^{(g-1)} \to \Theta$, one has:

(i) The composition $\gamma\circ\alpha = \varphi_\omega$, the canonical map of $C^{(g-1)}$.

(ii) The branch divisors $B\varphi_\omega = B\gamma \subset |\omega_C| = \mathbb{P}^{g-1*}$ are equal.

(iii) If $D = \Sigma p_i$ in $C^{(g-1)}$, and $\alpha(D) = L$ on Θ_{sm}, then $\gamma(L) = \varphi_\omega(D) = \overline{D}$ = the hyperplane in $\mathbb{P}^{g-1}$ spanned by the divisor D on C_ω. Projection from the closure of the graph of φ_ω, extends φ_ω to the map $(D,H) \mapsto H$ where $D = \Sigma p_i$ is any effective divisor of degree g-1, which is dominated by $H \cdot C_\omega$.

(iv) It follows [cf.A; ACGH, p.247] that $B\gamma$ = {hyperplanes H in $\mathbb{P}^{g-1}$ such that there are fewer than the maximum number of ways to choose a divisor D of degree g-1 on C dominated by $C_\omega \cdot H$} = $C_\omega{}^*$, the "dual variety" of hyperplanes tangent to C_ω. Then by double duality, C_ω = $B\gamma^*$ = {the variety of hyperplanes Λ in $|\omega_C|$ tangent to the branch divisor $B\gamma$}.

Singularities of Jacobian and Prym Gauss divisors

By analogy, we can ask whether the Prym canonical curve C_η is dual to the branch divisor of the Gauss map on the Prym theta divisor. Unfortunately the answer is essentially always no. To see why, consider first singularities of the "Gauss divisors" defined by a Jacobian Gauss map: $\gamma:\Theta ---->|\omega_C|$. If $\Lambda\subset|\omega_C|$ is a hyperplane in $|\omega_C|$, the inverse image $\gamma^*(\Lambda) = \Gamma\subset\Theta$ is the corresponding Gauss divisor. Thus when $B_\gamma\subset|\omega_C|$ is the branch divisor of γ, we expect that Γ will be singular over points where Λ is tangent to B_γ. For Jacobians, the set of hyperplanes $B_\gamma{}^*$ tangent to the divisor B_γ is the curve $B_\gamma{}^* = C_\omega$, i.e. there is only a one parameter family C_ω of hyperplanes tangent to the points of the hypersurface B_γ. Therefore, under the duality between $|\omega_C|$ and canonical space $\mathbb{P}^{g-1}$, a general hyperplane Λ_p corresponding to a point p on C_ω is tangent to B_γ in codimension one, hence Γ is singular in codimension one. This is a necessary condition for B_γ to be dual to a curve [SV5, Thm. part 3), Prop. following lemma 5], i.e. the Gauss divisors corresponding to points of that curve must be singular in codimension one. Hence the following result of Beauville and Debarre implies that for $g \geq 5$, the branch divisor of the general Prym Gauss map cannot be dual to a curve, (recalling that $\dim.\mathrm{sing}\Xi = p-6$ for general Prym varieties of dimension $p \geq 6$, and $\mathrm{sing}\Xi = \emptyset$ for general Pryms of dimension p = 4, 5).

Theorem 4.1. [BD2, p.619, Rmk.1] If (A,Θ) is a p.p.a.v. which is "simple" (no proper abelian subvarieties) and such that Θ is non singular in codimension ≤ 3, then all Gauss divisors Γ are normal and irreducible for $\dim(A) \geq 4$.

If we ask only whether the branch divisor of the Prym Gauss map can be dual to the Prym canonical curve, we have more precisely:

Theorem 4.2. [SV5] If C is non hyperelliptic of genus $g \geq 5$, then for any double cover, the Gauss divisor $\Gamma_{\overline{p}}$ corresponding to a general point $\overline{p}$ of the Prym canonical curve C_η is normal and irreducible. In particular, $(C_\eta)^* \not\subset B_\gamma$ = branch divisor of the Prym Gauss map γ_Ξ.

To see how to proceed with a Torelli argument for Pryms analogous to Andreotti's, given these results, we examine more closely the differences between Jacobian and Prym Gauss divisors. In the Jacobian case if $\Lambda_p \subset \mathbb{P}^{g-1*}$ is the hyperplane corresponding to a point p on C_ω, i.e. Λ_p = {all hyperplanes H in $\mathbb{P}^{g-1}$ such that p is in H}, and $\varphi_\omega: C^{(g-1)} \to |\omega_C|$ is the canonical map on $C^{(g-1)}$, then $C^{(g-1)} \supset \varphi_\omega^*(\Lambda_p)$ = {D : p is in $\overline{D}$}. Since D contains g-1 points of C_ω, and $\overline{D}$ cuts 2g-2 points of C_ω, there are two ways p can belong to $\overline{D}$, either p is a point of the divisor D, or p is residual to D in $\overline{D}\cdot C_\omega$. Thus if we define $C^{(g-1)} \supset \mathcal{D}_p$ = {D : p is in D}, and $\mathcal{D}'_p$ = {D : p is in $\overline{D}\cdot C_\omega$ -D}, then $\varphi_\omega^*(\Lambda_p) = \mathcal{D}_p \cup \mathcal{D}'_p \subset C^{(g-1)}$, and $\Gamma_p = \gamma^*(\Lambda_p) = \varphi_\omega(\mathcal{D}_p) \cup \varphi_\omega(\mathcal{D}'_p) \subset \Theta$, are reducible divisors, which explains why in particular they are singular in codimension one. To compare with the case of Pryms we will use the Abel parametrization $\varphi: X \to \Xi$ described in section 3.1 above to describe Gauss divisors of Ξ. By the remark of [BD1, p.202] that Grothendieck's solution of the Samuel conjecture implies Ξ is locally factorial when Ξ is non singular in codimension ≤ 3, all Weil divisors of

Ξ are Cartier when Cliff(C) ≥ 3, so we shall choose to ignore the singularities of Ξ in the remainder of this discussion.

Definition 4.3. For any point p of $\tilde{C}$, let $\mathcal{D}_p$ = {D in X: p is in D}, and $\mathcal{D}_{p'}$ = {D in X: p' is in D}, where p' = $\iota(p)$ is the point of $\tilde{C}$ conjugate to p under the involution $\iota:\tilde{C}\to\tilde{C}$ defined by the double cover $\pi:\tilde{C}\to C$.

Note that for each p, the restriction $\varphi:\mathcal{D}_p\to\Xi$ is a birational surjection, so in contrast to the Jacobian case $\varphi(\mathcal{D}_p)$ does not define a divisor in Ξ. However if we take two points p $\neq$ q of $\tilde{C}$, then $\varphi(\mathcal{D}_p) = \Xi = \varphi(\mathcal{D}_q)$, while $\mathcal{D}_p\cap\mathcal{D}_q$ is a proper intersection, so in some sense the image $\varphi(\mathcal{D}_p\cap\mathcal{D}_q)$ represents a self intersection of Ξ. Indeed Beauville showed this is true in homology, i.e. that the cohomology class of $\varphi(\mathcal{D}_p\cap\mathcal{D}_q)$ is that of a Gauss class, $c_1(\mathcal{O}_\Xi(\Xi))$. We can give a more geometric statement in terms of the "Abel Prym" map.

Definition 4.4. Given p on $\tilde{C}$, the Abel Prym map $a_p:\tilde{C}\to P_0\subset \mathrm{Pic}^0(\tilde{C})$, takes q to $a_p(q) = a(q,p) = (1-\iota)(q-p) = q-p-q'+p'$, thought of as the line bundle $\mathcal{O}(q-p-q'+p')$ on $\tilde{C}$. (This Abel Prym map is the composition of the usual Abel map $\tilde{\alpha}_p:\tilde{C}\to \mathrm{Pic}^0(\tilde{C})$ with base point p, with the projection $(1-\iota):\mathrm{Pic}^0(\tilde{C})\to P_0$.)

Lemma 4.5. [BD2, p.615] If $a(q,p') = a_{p'}(q) = (1-\iota)(q-p') = q-p'-q'+p$, and q $\neq$ p, p', then $\varphi(\mathcal{D}_p\cap\mathcal{D}_q) = \Xi\cap\Xi_{a(q,p')}$. (In the notation of [BD2], $a_{p'}(q)$ = [p,q].)

Note that the equality $\varphi(\mathcal{D}_p \cap \mathcal{D}_q) = \Xi \cap \Xi_{a(q,p')}$ cannot hold if q equals p or p', since in the first case the left hand side is improper (but not the right), and in the second case the right hand side is improper (but not the left). These two special cases, the first of which is explicitly treated in [BD2], are of particular interest for us. Since $a_{p'}(p') = 0$, if we let q approach p' in the equation $\varphi(\mathcal{D}_p \cap \mathcal{D}_q) = \Xi \cap \Xi_{a(q,p')}$, then we see in the limit that $\varphi(\mathcal{D}_p \cap \mathcal{D}_{p'})$ belongs to the Gauss linear system $|\vartheta_\Xi(\Xi)|$. In fact [SV5, part 1) of main theorem] if $\overline{p}$ is the point $\pi(p) = \pi(p')$ on the Prym canonical model C_η of C, then it follows that $\varphi(\mathcal{D}_p \cap \mathcal{D}_{p'}) = \Gamma_{\overline{p}}$ is the Gauss divisor corresponding to $\overline{p}$. (Containment in one direction is shown by a direct set theoretic calculation using Tjurin's description of the Prym canonical map, and then Beauville's homology calculation is invoked to conclude equality. A similar method proves Lemma 4.5 above.)

In particular, the "Prym canonical" Gauss divisors $\Gamma_{\overline{p}}$, are parametrized by intersections of sets $\mathcal{D}_p$, as opposed to unions of them in the case of the "canonical" Gauss divisors for Jacobians. (The intersections $(\mathcal{D}_p \cap \mathcal{D}_{p'})$ can be shown to be generically transverse [SV5] which then implies the normality of $\Gamma_{\overline{p}}$.) This makes it appear that the Gauss map and the Gauss divisors of a Prym theta divisor do not yield useful invariants for the recovery of $\tilde{C}$ or C in the usual way, so in the next section we alter our viewpoint slightly.

§5. Andreotti's birational formulation of Torelli

If we glance back at Andreotti's paper we see a subtle difference between his proof and the Gauss map version of it we sought to generalize in §4 above. In fact he stated his approach to the Torelli theorem without reference to the Gauss map as follows [A, p.801]: two smooth curves C_1, C_2 of genus g are birationally equivalent if and only if their symmetric products $C_1^{(g-1)}$ and $C_2^{(g-1)}$ are birationally equivalent. His argument for Torelli (in the non hyperelliptic case) may thus be given as follows: given two curves C_1, C_2 and their Jacobian varieties, $(J_1\ \Theta_1)$, (J_2, Θ_2), if the two theta divisors are isomorphic $\Theta_1 \cong \Theta_2$, then the Abel parametrizations imply the symmetric products are birationally equivalent $C_1^{(g-1)} \approx C_2^{(g-1)}$, and then he proved the branch divisors B_1, B_2 of the two corresponding canonical maps $\varphi_{\omega 1}$ and $\varphi_{\omega 2}$ of those symmetric products are isomorphic. Since $B_1 \cong C_1^*$ and $B_2 \cong C_2^*$ he concluded the curves C_1 and C_2 are isomorphic, from "double duality" for instance in characteristic zero. The key points are that a birational map $C_1^{(g-1)} \approx C_2^{(g-1)}$ carries regular sections of the canonical line bundle of $C_1^{(g-1)}$ to regular sections of the canonical line bundle of $C_2^{(g-1)}$, and that the branch divisor of a rational map is well defined.

For two double covers $\tilde{C}_1/C_1$, and $\tilde{C}_2/C_2$ (again with non hyperelliptic base curves) we know that if the theta divisors are isomorphic $\Xi_1 \cong \Xi_2$, then the divisor varieties X_1 and X_2, being generically $\mathbb{P}^1$ bundles over Ξ_1, Ξ_2 respectively, are birationally equivalent, and we can ask whether, or when, this implies the double covers are isomorphic. Of course the usual counterexamples to Prym Torelli are also counterexamples to such a general statement, but we can ask the question for $\tilde{C}_1/C_1$ and $\tilde{C}_2/C_2$ assuming $\mathrm{Cliff}(C_i) \geq 3$, and then we can prove the following result. (See 5.9 for a sketch of the proof.)

Theorem 5.1. [SV4, SV7] If $\tilde{C}_1/C_1$ and $\tilde{C}_2/C_2$ are connected étale double covers of smooth non hyperelliptic curves of genus $g \geq 3$ and both Prym theta divisors Ξ_i are non singular in codimension ≤ 3, and if $X_1 \cong X_2$ (biregular isomorphism), then $\tilde{C}_1/C_1 \cong \tilde{C}_2/C_2$.

Remark 5.2. The hypotheses hold in particular for curves C_i with $\mathrm{Cliff}(C_i) \geq 3$ (and hence of genus $g \geq 7$).

By theorem 5.1 the Donagi conjecture is equivalent to the assertion that if $\mathrm{Cliff}(C_i) \geq 3$, and $\Xi_1 \cong \Xi_2$, then $X_1 \cong X_2$. We can also formulate a "birational" version which would imply Donagi's conjecture as follows:

5.3. "Birational Donagi conjecture": If X_1, X_2 are divisor varieties associated to double covers $\tilde{C}_1/C_1$, $\tilde{C}_2/C_2$, and $\mathrm{Cliff}(C_i) \geq 3$ (hence $g \geq 7$), birational equivalence $X_1 \approx X_2$ implies biregular isomorphism $X_1 \cong X_2$.

Factorizing the Prym map

We may summarize the birational approach to Torelli as a factorization of the Torelli and Prym maps as follows. If $\mathfrak{J}(C) = (\mathrm{Pic}^0(C),\vartheta)$ is the Jacobian variety of C, we may factor the Torelli map $C \mapsto \mathfrak{J}(C)$ through the space $\mathfrak{C}$ parametrizing symmetric products of curves, by defining $\alpha(C) = C^{(g-1)}$, and $\beta(C^{(g-1)}) = (\mathrm{Pic}^0(C),\Theta) = \mathrm{Alb}(C^{(g-1)})$ [note the theta divisor Θ is recoverable as the image of the Albanese map $C^{(g-1)} \rightarrow \mathrm{Alb}(C^{(g-1)})$], yielding a commutative diagram as follows.

$$
\begin{array}{ccc}
 & \alpha & \\
\mathfrak{M}_g & \rightarrow & \mathfrak{C} \\
 \mathfrak{J} & \diagdown \; \diagup \, \beta & \\
 & \mathfrak{A}_g &
\end{array}
$$

Then Andreotti's construction, recovering the curve C from the branch divisor of the canonical map on $C^{(g-1)}$, shows that α is injective, but more subtly, his proof also shows that if we mod out by birational equivalence in the space $\mathfrak{C}$, the composition $\tilde{\alpha}$ is still injective. Since two symmetric products which correspond to birational theta divisors are themselves birational the induced map $\tilde{\beta}$ is also injective, hence Torelli follows from the diagram below. (Note that the Albanese varieties of two birational symmetric products are isomorphic, so $\tilde{\beta}$ is well defined. Alternatively, since $\tilde{\alpha}$ is injective, and like α it is also surjective, the sets $\mathfrak{C}$ and $\mathfrak{C}/\approx$ are bijectively equivalent, so $\tilde{\beta}$ may be defined to be β, or the composition $\mathfrak{J}\circ\tilde{\alpha}^{-1}$.)

$$\begin{array}{ccc} & \tilde{\alpha} & \\ \mathfrak{M}_g & \rightarrow & \mathfrak{C}/\approx \\ & \mathfrak{J} \quad \uparrow \tilde{\beta} & \\ & \mathfrak{A}_g & \end{array}$$

In the case of Pryms then we try to imitate this approach by factoring $\mathfrak{P}$ through the space $\mathfrak{X}$ parametrizing the divisor varieties X associated to double covers of curves of genus g. For simplicity let $\mathcal{R}_g\{\text{Cliff} \geq 1\}$ denote the space of double covers of non hyperelliptic curves C (since then the divisor variety X is irreducible).

$$\begin{array}{ccc} & \alpha & \\ \mathcal{R}_g\{\text{Cliff} \geq 1\} & \rightarrow & \mathfrak{X} \\ & \mathfrak{P} \quad \uparrow \beta & \\ & \mathfrak{A}_{g-1} & \end{array}$$

Then Theorem 5.1 above implies that the restriction of the map $\alpha: \tilde{C}/C \mapsto X$ to the set $\mathcal{R}_g\{\text{Cliff} \geq 3\}$ is injective. If $\beta(X)$ = the Albanese variety of X (in the sense of the universal object for morphisms from X to abelian varieties), then it is again true that two X's on which β takes the same value (P, Ξ) are birationally equivalent (to $\Xi \times \mathbb{P}^1$), but we do not know how to show that α induces an injection when composed with the map $\mathcal{X} \to \mathcal{X}/\approx$ taking an isomorphism class of X's to the corresponding birational equivalence class.
We can now however replace $\mathcal{P}$ by the map β, and the Prym Torelli problem becomes that of classifying "Prym $\mathbb{P}^1$ bundles" over Prym theta divisors. If g = 3, we have shown in [SV4] that the Prym $\mathbb{P}^1$ bundles for non hyperelliptic C, are all those stable even $\mathbb{P}^1$ bundles over the smooth genus two curve Ξ, with smooth "Narasimhan - Ramanan" invariant (see [NR], or definition below).

The norm map on divisors

We can make the analogy with Andreotti's argument for Jacobians more precise as follows. Recalling from the discussion above that the canonical line bundle for the symmetric product $C^{(g-1)}$ is represented by reducible divisors, i.e. that $\omega_{C(g-1)} = \mathcal{O}_{C(g-1)}(\mathcal{D}_p+\mathcal{D}'_p)$, where $\mathcal{D}_p$ and $\mathcal{D}'_p$ are defined in the discussion preceding Definition 4.3 above, we define the analogous line bundle $\mathcal{O}_X(\mathcal{D}_p+\mathcal{D}_p')$, denoted $\mathcal{O}_X(1)$, for a double cover $\tilde{C}/C$, where C is non hyperelliptic of genus $g \geq 3$. If $h:X \to |\omega_C|$ is the restriction to X of the norm map $\text{Nm}:\tilde{C}^{(2g-2)} \to C^{(2g-2)}$ on divisors, then [SV4; SV6; cf. IP, Lemmas 2.2, 2.5]:

Proposition 5.4.

(1) $h^*(\mathcal{O}_{|\omega_C|}(1)) = \mathcal{O}_X(1)$.

(2)(i) $B_h = C_{\omega}^*$; **(ii)** $h^*(\Lambda_{\overline{p}}) = \mathcal{D}_p + \mathcal{D}_{p'}$, so that h determines $\tilde{C}/C$.

(3) If K_Ξ and K_X are canonical divisors on Ξ, X respectively, then $\mathcal{O}_X(1) \cong \mathcal{O}_X(2\varphi^*(K_\Xi) - K_X)$.

Thus the map whose branch divisor we want to recover is determined by a linear subsystem of $|2\varphi^* K_\Xi - K_X|$, a combination of the canonical bundles of X and Ξ.

Proof of (3) (for all, i.e. possibly hyperelliptic, C with $g \geq 3$):
We will identify the canonical bundle of X and this formula will result. We use the realization of $X \subset \tilde{C}^{(2g-2)}$ as a connected component of the norm preimage of $|\omega_C| \subset C^{(2g-2)}$, adjunction, and Macdonald's formula for the canonical class of a symmetric product. Since X is a local complete intersection in the smooth variety $\tilde{C}^{(2g-2)}$, the canonical divisor class K_X is Cartier and is given by $\mathcal{O}_X(K_X) \cong \mathcal{O}_X(\tilde{K}) \otimes \Lambda^{g-1}(\tilde{N})$, where $\tilde{K}$ denotes the canonical class of $\tilde{C}^{(2g-2)}$ and $\tilde{N}$ denotes the "normal bundle" of X in $\tilde{C}^{(2g-2)}$. We will show (1) $\mathcal{O}_X(\tilde{K}) \cong \varphi^*(\mathcal{O}_\Xi(2K_\Xi))$ and (2) $\Lambda^{g-1}(\tilde{N}) \cong \mathcal{O}_X(-1)$; then we get the formula $\mathcal{O}_X(K_X) \cong \varphi^*(\mathcal{O}_\Xi(2K_\Xi)) \otimes \mathcal{O}_X(-1)$, which is equivalent to the stated formula $\mathcal{O}_X(1) \cong \mathcal{O}_X(2\varphi^*(K_\Xi) - K_X)$.
For (1), use $\tilde{K} = K_{\tilde{C}(2g-2)} = \tilde{\alpha}^*(K_{\tilde{\Theta}})$, which follows since $\tilde{C}^{(2g-2)} \to \tilde{\Theta}$ is a small resolution for C non hyperelliptic, and is generically a product of the standard resolution of an ordinary rank 3 double point, along the singular locus for C hyperelliptic. Hence $\mathcal{O}_X(\tilde{K}) = \mathcal{O}(\tilde{K}|_X) = \tilde{\alpha}^*\mathcal{O}(K_{\tilde{\Theta}})|_X =$

$\varphi^*(\mathcal{O}(K_{\tilde{\Theta}}|_{\Xi})) = \varphi^*(\mathcal{O}_{\Xi}(2K_{\Xi}))$ (since $\mathcal{O}(K_{\tilde{\Theta}}) = \mathcal{O}_{\tilde{\Theta}}(\tilde{\Theta}) = \mathcal{O}_{\tilde{J}}(\tilde{\Theta})|_{\tilde{\Theta}}$ and $\mathcal{O}_{\tilde{J}}(\tilde{\Theta})|_P = \mathcal{O}_P(2\Xi)$, hence $\mathcal{O}(K_{\tilde{\Theta}})|_{\Xi} = \mathcal{O}_{\tilde{J}}(\tilde{\Theta})|_{\Xi} = \mathcal{O}_P(2\Xi)|_{\Xi} = \mathcal{O}_{\Xi}(2\Xi) = \mathcal{O}(2K_{\Xi})$).

For (2), recall first the construction of $\tilde{N}$. Restricting the sheaf $\tilde{\Omega}^1$ of Kähler differentials for $\tilde{C}^{(2g-2)}$ to X yields an exact sequence: $0 \to \tilde{I}/\tilde{I}^2 \to \tilde{\Omega}^1|_X \to \Omega^1{}_X \to 0$, in which $\tilde{I} \subset \tilde{\mathcal{O}}$ is the ideal sheaf of $X \subset \tilde{C}^{(2g-2)}$ in the structure sheaf of $\tilde{C}^{(2g-2)}$ and then $\tilde{I}/\tilde{I}^2$ is, by definition, the "conormal bundle" $\tilde{N}^*$ of X in $\tilde{C}^{(2g-2)}$ and is locally free of rank g-1, the codimension of X. ($\tilde{N}$ is then $(\tilde{I}/\tilde{I}^2)^*$.) The mapping from $\tilde{I}/\tilde{I}^2$ to $\tilde{\Omega}^1|_X$ is $f \mapsto df|_X$. (Notice that if X is singular at a point x, then the fibre $(I/I^2)|_x$ has dimension g-1, but the linear map $(\tilde{I}/\tilde{I}^2)|_x \to \tilde{\Omega}^1|_x$ there is not injective, and correspondingly, the fibre $(\Omega^1{}_X)|_x$ has dimension greater than g-1 at such a point.)

We claim that $\tilde{N}$ is pulled back by Nm from the normal bundle $N = N_{|\omega_C|/C^{(2g-2)}}$. For this, it suffices to observe that $\tilde{N}^*$ is indeed pulled back from the conormal bundle N^* of $|\omega_C|$ in $C^{(2g-2)}$. Since the l.c.i. $X \subset \tilde{C}^{(2g-2)}$ is (a connected component of) the preimage scheme of the l.c.i $|\omega_C| \subset C^{(2g-2)}$, the ideal I of $|\omega_C|$ in $C^{(2g-2)}$ pulls back to the ideal $\tilde{I}$ of X in $\tilde{C}^{(2g-2)}$ and then $I/I^2 = N^*$ pulls back to $\tilde{I}/\tilde{I}^2 = \tilde{N}^*$. It follows that $\tilde{N}$ is the pullback of N and $\wedge^{g-1}(\tilde{N})$ is the pullback of $\wedge^{g-1}(N)$.

We need to use the canonical class of a symmetric product. The formula for the chern polynomial $c_t = \Sigma c_i t^i$ of the tangent bundle of $C^{(d)}$ is [cf. ACGH, pp. 322-323]:

$$c_t(C^{(d)}) = (1+tx)^{d-g+1}e^{-t\vartheta/(1+tx)},$$

where x is the ample class on $C^{(d)}$ and ϑ is pulled back from the polarizing class on $\mathrm{Pic}^d(C)$. (Alternatively [ACGH, p. 339], the chern

character is: $g-1 + ((d-g+1) - \vartheta)e^x$.) Expanding we have:
$c_t = (1+(d-g+1)xt+\ldots)e^{-\vartheta t(1-xt+\ldots)} = (1+(d-g+1)xt+\ldots)(1-\vartheta t+\ldots)$
$= 1 + ((d-g+1)x - \vartheta)t + \ldots$, hence in particular:

$$c_1(C^{(d)}) = (d-g+1)x - \vartheta.$$

Macdonald determined in [Mac] all the chern classes of $C^{(d)}$; for specifically the canonical class, see [Mac, p. 334, before (14.10) (from (14.5) on p. 332)]. Note that in case $d = g-1$, and in no other case, the first chern class of $C^{(g-1)}$ is pulled back from a class on $\mathrm{Pic}^{g-1}(C)$. For the case $d = 2g-2$, we have $c_1(C^{(2g-2)}) = (g-1)x - \vartheta$. Now it is easy to figure out the top exterior power $\Lambda^{g-1}(N)$ of N; namely, by adjunction on $|\omega_C| \cong \mathbb{P}^{g-1}$, we have $K_{\mathbb{P}^{g-1}} = ((K_C(2g-2))|_{\mathbb{P}^{g-1}}) \otimes \Lambda^{g-1}(N)$. But (i) $K_{\mathbb{P}^{g-1}} \cong \mathcal{O}(-g)$ and (ii) $(K_C(2g-2))|_{\mathbb{P}^{g-1}} \cong \mathcal{O}(g-1)$ since $c_1(C^{(2g-2)}) = (g-1)x - \vartheta$, ϑ restricts trivially to $|\omega_C|$ (because $|\omega_C|$ maps to a point in $\mathrm{Pic}^{2g-2}(C)$), and the line bundle $(K_C(2g-2))|_{\mathbb{P}^{g-1}}$ on projective space is determined by its chern class. Thus, on $\mathbb{P}^{g-1}$ we get $\mathcal{O}(-g) \cong \mathcal{O}(g-1) \otimes \Lambda^{g-1}(N)$, and hence $\Lambda^{g-1}(N) \cong \mathcal{O}(-1)$. **Q.E.D.**

5.5. Conjectures.

(i) The map h is determined by a complete linear system if C is non hyperelliptic.

(ii) If $\mathrm{Cliff}(C) \geq 3$, the branch divisor B_h is determined up to isomorphism by the birational equivalence class of X.

Remark 5.6. If $g = 3$, both conjectures hold. (Conjecture 5.5.(ii) holds vacuously since both the hypothesis and conclusion fail.)

Method of proof of Theorem 5.1

We will adapt to the generic $\mathbb{P}^1$ bundle $\varphi: X \to \Xi$, the invariant introduced by Narasimhan and Ramanan in their paper [NR] in which they classify semi stable rank 2 vector bundles over genus two curves up to S-equivalence, (and which has since been generalized to analyze semi stable vector bundles of all ranks over curves of genus ≥ 2). To the best of our knowledge, the first place this type of argument is used to recover a curve from the data of the set of its effective Poincare or Picard sheaves, is in [Ke1, p.16; cf. Ke2, Cor.4.4, p.253].

To give the flavor of the proof of Thm. 5.1 in a simpler case, we discuss an analog of Kempf's results from our more geometric point of view. The cases Kempf dealt with were higher derived sheaves of Poincare bundles parametrizing line bundles of negative degree on a curve C, yielding high rank vector bundles dual to those parametrizing sections of line bundles on C of degree $> 2g-2$. He defined a Narasimhan Ramanan invariant for these sheaves which recovers the curve C. We will look instead at line bundles of degree g+1 on C, and show that the corresponding Poincare bundles with appropriate chern class, push down to reflexive rank 2 sheaves on Ξ, which although not vector bundels, also have an NR invariant which recovers the curve C.

Effective Poincare bundles on Jacobian varieties of curves

Let C be a curve, d be an integer, and denote $Pic^d(C)$ by J^d. A line bundle $\mathcal{L}$ on $C \times J^d$ induces by restriction two maps: $J^d \to Pic(C)$ taking $\mathcal{L}$ to $\mathcal{L}|C \times \{L\}$, and $C \to Pic(J^d)$ taking q to $\mathcal{L}|\{q\} \times J^d$. $\mathcal{L}$ is called a Poincare bundle on $C \times J^d$ if the first induced map is the identity map

$J^d \to Pic^d(C)$. We will call $\mathcal{L}$ a standard Poincare bundle if the second induced map takes $C \to Pic^{\vartheta}(J^d)$, i.e. if the chern class of each restriction $\mathcal{L}|\{q\}\times J^d$ has cohomology class ϑ = the canonical polarization class on J^d. If $\mathcal{L}$ is effective then its generic restriction to $C\times\{L\}$ will also be effective, hence $\mathcal{L}$ cannot be effective for $1 \le d < g = g(C)$. As a preliminary question, which introduces the basic concepts, we ask for effective Poincare bundles of degree g. We will see that in this case the construction does not carry enough information to determine C.

Effective Poincare bundles on $C\times J^g$

Proposition 5.7: The only effective standard Poincare bundle on $C\times J^g$ is the one whose second restriction map is the standard Abel map $\alpha: C \to Pic^{\vartheta}(J^g)$ taking p to $\mathcal{O}(\Theta_p)$ where $\Theta_p \subset Pic^g(C)$ is the translate of $\Theta \subset Pic^{g-1}(C)$ by the line bundle $\mathcal{O}_C(p)$.

Proof: First we show how to construct an effective standard Poincare bundle [Ke4, lemma 18.1, p.155] of degree g. We want to produce a natural effective divisor on $C\times J^g$. Consider first the product $C\times C^{(g)}$ and the natural (irreducible) incidence divisor $C\times C^{(g)} \supset I = \{(q,D): q \text{ is in } D\} = \cup_q \{q\}\times\mathcal{D}_q$, where $\mathcal{D}_q = \{D: q \text{ is in } D\}$. Consider the birational Abel map $\varphi: C^{(g)} \to J^g$, the associated birational product map $1\times\varphi: C\times C^{(g)} \to C\times J^g$, and the image $(1\times\varphi)(I) = S \subset C\times J^g$, of the incidence divisor. Note $S = \{(q,L): |L-q| \neq \varnothing\} = \{(q,L): |L-q| \text{ is in } \Theta\} = \cup_q \{q\}\times\Theta_q$, where Θ_q = the translate of Θ by $\mathcal{O}_C(q)$. We claim $\mathcal{L} = \mathcal{O}(S)$ is the only effective standard Poincare bundle on $C\times J^g$. First of all $\mathcal{L}$ is effective since $S \ge 0$. $\mathcal{L}$ is a Poincare bundle since for L general in J^g, $\mathcal{L}$ restricts on $C\times\{L\}$ to the line bundle on C defined by the restriction of the divisor S. But for

L general $|L| = \{D\}$ contains only one effective divisor, so (q,L) belongs to S if and only if q is one of the points of $|L| = D$. Thus $S|C\times\{L\} = D = |L|$, and thus $\mathcal{L}|C\times\{L\} = L$. The induced morphism $J^d \to Pic^d(C)$ is thus the identity on an open set and hence the identity everywhere by continuity. $\mathcal{L}$ is a standard Poincare bundle since for q in C, the restriction of $S = \cup_q \{q\}\times\Theta_q$ to $\{q\}\times J^g$ is Θ_q, whose chern class is ϑ. We claim there are no other effective standard Poincare bundles on $C\times J^g$. First of all if $\mathcal{L} = \mathcal{O}(S)$ is the Poincare bundle defined above, all other Poincare bundles on $C\times J^g$ have form $\mathcal{L}\otimes\mu^*(\tau)$ for some τ in $Pic(J^g)$. Moreover since the restriction of $\mu^*(\tau)$ to a cross section of form $\{q\}\times J^g$ is isomorphic to τ itself, all other <u>standard</u> Poincare bundles have form $\mathcal{L}\otimes\mu^*(\tau)$ for τ in $Pic^0(J^g)$. Since $J^0 \cong Pic^0(J^g)$ via $c \mapsto \mathcal{O}(\Theta_c-\Theta)$, we must show the only effective bundle of form $\mathcal{L}\otimes\mu^*(\mathcal{O}(\Theta_c-\Theta))$ for c in J^0 is $\mathcal{L}$, i.e. we must have $c = 0$. Let $\tilde{S}$ be an effective divisor linearly equivalent to $S+\mu^*(\Theta_c-\Theta)$. Then the restriction of $\tilde{S}$ to a general fiber of form $C\times\{L\}$ must be an effective divisor in the linear system $|L|$, which in fact contains only one effective divisor. Thus the part of $\tilde{S}\subset C\times J^g$ which surjects onto J^g via the projection $\mu: C\times J^g \to J^g$ is uniquely determined and must equal S. Thus $\tilde{S} = S + \mu^*(E)$ for some effective divisor E of chern class zero on J^g. Then $E = 0$,which implies that $c = 0$, and $\tilde{S} = S$. **QED for Prop 5.7.**

Next we see that for degree g+1, the set of effective standard Poincare bundles contains more information.

Effective Poincare bundles on $C\times J^{g+1}$

For each point p of C, define an Abel map $\varphi_p: C \to Pic^{\vartheta}(J^{g+1})$ taking q to

$\varphi_p(q) = \mathcal{O}_{J^{g+1}}(\Theta_{p+q})$. We will show the only effective standard Poincare bundles on $C \times J^{g+1}$ are those for which the "second restriction map" is one of the Abel maps $\varphi_p: C \to \mathrm{Pic}^\vartheta(J^{g+1})$. First we will define such an effective bundle $\mathcal{L}_p$ for each point p of C as follows. Consider the set $C^{(g+1)} \supset \mathcal{D}_p$ = {D in $C^{(g+1)}$: p is in D}, and the restricted incidence divisor $C \times \mathcal{D}_p \supset I_p$ = {(q,D): q is in D and p is in D}.

Note: We do <u>not</u> say in the previous definition that p+q is in D. Then $\mathcal{D}_p$ is irreducible and if $\varphi: C^{(g+1)} \to J^{g+1}$ is the Abel map in degree g+1, the restricted Abel map $\varphi_p: \mathcal{D}_p \to J^{g+1}$ is birational, and the restricted product map $(1 \times \varphi_p): C \times \mathcal{D}_p \to C \times J^{g+1}$ is also birational. Define the effective divisor $S_p = (1 \times \varphi_p)(I_p) \subset C \times J^{g+1}$, and the line bundle $\mathcal{L}_p = \mathcal{O}(S_p)$.

Proposition 5.8: The second restriction map for $\mathcal{L}_p$ is the Abel map $\varphi_p: C \to \mathrm{Pic}^\vartheta(J^{g+1})$, and the bundles $\{\mathcal{L}_p$: p in C} are the only effective standard Poincare bundles on $C \times J^{g+1}$.

Proof: Note that the restricted incidence variety $I_p = \cup_q \{q\} \times (\mathcal{D}_p \cap \mathcal{D}_q)$, is reducible this time since it contains the component $\{p\} \times \mathcal{D}_p$. The image divisor S_p also is reducible and equals $S_p = \cup_q \{q\} \times \varphi(\mathcal{D}_p \cap \mathcal{D}_q) = (\cup_q \{q\} \times \Theta_{p+q}) + (\{p\} \times J^{g+1})$. This shows $\mathcal{L}_p$ is a standard effective Poincare bundle on $C \times J^{g+1}$ inducing the map $\varphi_p: C \to \mathrm{Pic}^\vartheta(J^{g+1})$ where $\varphi_p(q) = \mathcal{O}_{J^{g+1}}(\Theta_{p+q})$. Fix p in C. If $\mathcal{L}$ is any other effective standard Poincare bundle on $C \times J^{g+1}$ then $\mathcal{L} \cong \mathcal{L}_p \otimes \mu^*(\tau)$ for some τ in $\mathrm{Pic}^0(J^{g+1})$, where $\mu: C \times J^{g+1}$ is projection. This means $\mathcal{L} \cong \mathcal{L}_p \otimes \mu^*(\mathcal{O}(\Theta_c - \Theta)) = \mathcal{O}(S_p + \mu^*(\Theta_c - \Theta)) = \mathcal{L}_c$, for c some point in J^0. Assume $\mathcal{L}_c$ is any effective standard Poincare bundle, so that $\mathcal{L}_c$ =

$\mathcal{O}(S_p+\mu^*(\Theta_c-\Theta)) \cong \mathcal{O}(S_c)$ for some effective divisor S_c. Then for general L in J^{g+1}, S_c meets $C\times\{L\}$ in an effective divisor belonging to $|L|$ but this time there is a pencil of such divisors. So consider the other restriction. I.e. for general q in C, S_c meets $\{q\}\times J^{g+1}$ in an effective divisor linearly equivalent to $\Theta_{p+q} + \Theta_c - \Theta$. Since ϑ is a principal polarization, every line bundle in $\mathrm{Pic}^{\vartheta}(J^{g+1})$ has precisely one section, hence there is only one such divisor, namely Θ_{c+p+q} by the theorem of the square. So if we define an effective divisor M_c on $C\times J^{g+1}$ by $M_c = \cup_q \{q\}\times\Theta_{c+p+q}$, then S_c can only differ from M_c by some effective divisor pulled back from C, via the projection $f:C\times J^{g+1}\to C$. To see what this pullback is, restrict $\mathcal{L}_c\otimes\mathcal{O}(-M_c) = \mathcal{O}(S_p+\mu^*(\Theta_c-\Theta))\otimes\mathcal{O}(-M_c)$ over $C\times\{L\}$ for some general L. Restricting the divisor $S_p+\mu^*(\Theta_c-\Theta)-M_c$ to $C\times\{L\}$ gives $\{(q,L)$: L belongs to $\varphi(\mathcal{D}_p\cap\mathcal{D}_q)\}$ + $\{(q,L)$: L is in $\Theta_c\}$ - $\{(q,L)$: L is in $\Theta\}$ - $\{(q,L)$:L is in $\Theta_{c+p+q}\}$, which via the isomorphism $C\times\{L\} \cong C$, gives the following divisor on C,

$\{q$: q = p or L is in $\Theta_{q+p}\}$ + $\{q$: L is in $\Theta_c\}$ - $\{q$: L is in $\Theta\}$ - $\{q$:L is in $\Theta_{c+p+q}\}$.

Since a general L will not belong to either Θ or Θ_c, this is just $\{q$: q = p or L is in $\Theta_{q+p}\}$ - $\{q$: L is in $\Theta_{c+p+q}\}$, which equals $\{q$: q = p or -q is in $\Theta_{p-L}\}$ - $\{q$: -q is in $\Theta_{c+p-L}\}$ = $\{q$: q = p or q is in $(-1)^*(\Theta_{(p-L)} - \Theta_{c+(p-L)})\}$.

If $\alpha:C\to\mathrm{Pic}^1(C)$ is the canonical Abel map taking q to $\mathcal{O}(q)$, this is the divisor $p + \alpha^*((-1)^*(\Theta_{(p-L)} - \Theta_{c+(p-L)}))$. By Riemann's proof of Jacobi inversion, and using symmetry of an appropriate translate of the theta divisor, $\alpha^*((-1)^*(\Theta_{(p-L)} - \Theta_{c+(p-L)}))$ is a divisor representing the line bundle c in $\mathrm{Pic}^0(C)$.

Hence $\mathcal{L}_c \otimes \mathcal{O}(-M_c) = f^*(\mathcal{O}(p) \otimes c)$, and by the argument above, the effective divisor $S_c - M_c$ is the pullback of an effective divisor on C representing $\mathcal{O}(p) \otimes c$. In particular the line bundle $\mathcal{O}(p) \otimes c$ on C is effective of degree one, which happens if and only if c in $\mathrm{Pic}^0(C)$ has form $\mathcal{O}(q-p)$ for some q on C, if and only if c belongs to the Abel curve $\alpha_p(C)$, where $\alpha_p: C \to \mathrm{Pic}^0(C)$ takes $q \mapsto \mathcal{O}(q-p)$. Thus the only effective Poincare bundles have form $\mathcal{L}_p \otimes \mu^*(\Theta_{q-p} - \Theta)$, which we claim is in fact $\mathcal{L}_q$. To see this note that by restricting to a general fiber of form $\{r\} \times J^{g+1}$, we get $\mathcal{L}_q \otimes \mathcal{L}_p^* \cong \mu^*(\Theta_{q+p} - \Theta_{p+r}) \cong \mu^*(\Theta_c - \Theta)$ if and only if $c = (q+r) - (p+r) = q-p$, (using the theorem of the square). Thus $\mathcal{L}_q \cong \mathcal{L}_p \otimes \mu^*(\Theta_{q-p} - \Theta)$, as claimed. **QED for Prop. 5.8**

This construction almost recovers the curve C from the divisor variety $C^{(g+1)}$ as follows. The variety $C^{(g+1)}$ recovers the Abel map $\varphi: C^{(g+1)} \to J^{g+1}$ with generic fibers $\cong \mathbb{P}^1$, as the Albanese map of $C^{(g+1)}$, in particular it recovers the Picard variety J^{g+1} as the image of the Albanese map. This generic $\mathbb{P}^1$ fibration then determines a family of "rank 2 sheaves" $\mathcal{E}$ (generically rank 2 vector bundles) on J^{g+1}, such that $C^{(g+1)} \cong \mathbb{P}(\mathcal{E})$ at least generically. Fixing $c_1(\mathcal{E}) = \vartheta$, the corresponding subfamily of these sheaves are those of form $\mathcal{E} \cong \varphi_*(\mathcal{L})$ for $\mathcal{L}$ a standard Poincare line bundle as above. We claim the "Narasimhan Ramanan invariant" $NR(\mathcal{E}) = \{\tau \text{ in } \mathrm{Pic}^0(J^{g+1}): h^0(\mathcal{E} \otimes \tau) \neq 0\}$ of this subfamily of sheaves determines C. For $\mathcal{L} = \mathcal{L}_p$, the computation above shows that $NR(\mathcal{E}) \cong \{c \text{ in } \mathrm{Pic}^0(C): h^0(\mathcal{L}_c) \neq 0\} = \alpha_p(C)$, is determined by $C^{(g+1)}$ and ϑ up to translation in $\mathrm{Pic}^0(C)$. Thus the data of the variety $C^{(g+1)}$ plus the cohomology class ϑ in

$H^2(J^{g+1}, \mathbb{Z})$ determines C. Since J^{g+1} is the Albanese variety of $C^{(g+1)}$, this would follow from classical Torelli, but we will see that this proof can be generalized, in a stronger form, to the case of Pryms where the classical Prym Torelli problem is still open. I.e. next we will show that in the case of Pryms, with suitable restrictions on C, we can recover the double cover $\tilde{C}/C$ just from the divisor variety X.

Effective Poincare bundles on Prym theta divisors

Given a suitable double cover $\tilde{C}/C$, our goal is to recover the Abel Prym curve $a_p(\tilde{C}) \subset P_0$, up to translation, from the divisor variety X, as any one of the Narasimhan Ramanan invariants associated to X. I.e. We claim X determines Ξ and the chern class $\xi = c_1(\mathcal{O}_\Xi(\Xi))$ as well as a distinguished family of rank 2 sheaves on Ξ, and that the corresponding Narasimhan Ramanan invariant of any one of these sheaves is an Abel Prym model of $\tilde{C}$, hence determines the double cover $\pi:\tilde{C}\to C$.

More precisely:

Theorem 5.9 [SV7]: If C is a smooth curve of genus $g \geq 3$, and $\pi:\tilde{C}\to C$ a connected étale double cover such that either Ξ is smooth or $\dim.\mathrm{sing}\Xi \leq p-5$, where $p = g(C)-1$ is the dimension of the Prym variety $(P(\tilde{C}/C), \Xi)$, then X determines the double cover $\pi:\tilde{C}\to C$.

Outline of proof:

(1) The Abel map $\varphi:X\to\Xi\subset P$ is, up to isomorphism, the Albanese map of X, hence X determines φ, Ξ, P, and $c_1(\mathcal{O}_\Xi(\Xi)) = \xi$.

(2) There exists a reflexive, coherent, rank 2 sheaf $\mathcal{E}$ on Ξ such that $X = \mathbb{P}(\mathcal{E})$ and $c_1(\mathcal{E}) = \xi = c_1(\mathcal{O}_\Xi(\Xi))$, at least over Ξ_{sm}, the smooth points of Ξ.

(3) (i) Any sheaf $\mathcal{E}'$ as in (2) has form $\mathcal{E}' \cong \mathcal{E}\otimes\tau$ for some τ in $Pic^0(\Xi)$.

(ii) Thus for $\mathcal{E}$ any sheaf as in (2), $NR(\mathcal{E}) = \{\tau$ in $Pic^0(\Xi)$ such that $h^0(\mathcal{E}\otimes\tau) \neq 0\}$ is determined, up to translation in $Pic^0(\Xi)$, by X.

(4) For each point p of $\tilde{C}$, there exists a sheaf $\mathcal{E}_p$ satisfying the properties in (2), and such that the composition $P_0 \cong Pic^0(P_0) \cong Pic^0(\Xi)$ of the polarization isomorphism with restriction (which takes c in P_0 to $\mathcal{O}_\Xi(\Xi_c-\Xi)$), carries the Abel Prym curve $a_p(\tilde{C})$ isomorphically to $NR(\mathcal{E}_p)$.

(5) Since X determines the Abel Prym curve $a_p(\tilde{C})$ which has a unique (-1) symmetry up to translation, X determines $\tilde{C}$ and the involution ι hence also the double cover $\tilde{C}/C$.

Sketch of proof of (4). We claim first that sheaves $\mathcal{E}$ as in (2) above arise as $\mu_*(\mathcal{L})$ for suitable Poincare bundles $\mathcal{L}$ on $\tilde{C}\times\Xi$. Consider the universal (rational) projective map $\gamma:\tilde{C}\times\Xi$--->X where $\gamma(p,L)$ = the unique divisor in $|L-p|$, and the two projections $f:\tilde{C}\times\Xi\to\tilde{C}$ and $\mu:\tilde{C}\times\Xi\to\Xi$. For each point p of $\tilde{C}$, if $\mathcal{D}_p$ = {D in X such that p belongs to D} $\subset$ X, and I_p = {(q,D): q is in D and p is in D}$\subset$ $\tilde{C}\times\mathcal{D}_p$ is the corresponding incidence variety, and S_p is the image of I_p in $\tilde{C}\times\Xi$, one checks that $\gamma^*(\mathcal{D}_p) = S_p$ $= \cup\{\{q\}\times\varphi(\mathcal{D}_p\cap\mathcal{D}_q)$, for all q in $\tilde{C}\}$ =
$\cup\{\{q\}\times\Xi\cap\Xi_{a(q,p')}$ for all q in $\tilde{C}\}$. Moreover, $\mathcal{O}(S_p) = \mathcal{L}_p$ is an effective Poincare bundle for $\tilde{C}\times\Xi$, and $\mathcal{E}_p = \mu_*(\mathcal{L}_p)$ is a rank 2 sheaf with the properties in (2) above. Then the projection formula implies that $NR(\mathcal{E}_p) = \{\tau$ in $Pic^0(\Xi)$: $h^0(\mathcal{L}_p\otimes\mu^*(\tau)) \neq 0\}$. Since elements of $Pic^0(\Xi)$ have form $\mathcal{O}_\Xi(\Xi_c-\Xi)$, it is enough to classify those c in $P_0\subset Pic^0(\tilde{C})$ such

that $\mathcal{L}_p \otimes \mu^*(\mathcal{O}_\Xi(\Xi_c - \Xi)) = \mathcal{L}_c$ is effective on $\tilde{C}\times\Xi$. Using the projection $f:\tilde{C}\times\Xi \to \tilde{C}$, it suffices to compute those c such that $f_*(\mathcal{L}_c)$ is effective on $\tilde{C}$. If $M_c = \mathcal{O}_{\tilde{C}\times\Xi}(\cup_q \{q\}\times\Xi_{c+a(q,p')})$, then one calculates that

(i) $\mathcal{L}_c = M_c \otimes f^*(c) \otimes f^*(\mathcal{O}_{\tilde{C}}(p-p'))$, hence

(ii) $f_*(\mathcal{L}_c) = f_*(M_c) \otimes c \otimes \mathcal{O}_{\tilde{C}}(p-p')$.

We claim if c is not a point of $a_p(\tilde{C})$ then $f_*(M_c)$ is trivial. I.e. if c is not a point of $a_p(\tilde{C})$, equivalently -c is not a point of $a_{p'}(\tilde{C})$, then the subscript $c+a(q,p')$ on Ξ in $M_c = \mathcal{O}_{\tilde{C}\times\Xi}(\cup_q \{q\}\times\Xi_{c+a(q,p')})$ is never zero, and thus the effective divisor $(\cup_q \{q\}\times\Xi_{c+a(q,p')})$ on $\tilde{C}\times\Xi$ restricts to a proper effective divisor on every fiber of f. Hence the pushdown has a never vanishing section, i.e. $f_*(M_c)$ is trivial. Thus for c not in $a_p(\tilde{C})$, $f_*(\mathcal{L}_c) \cong c \otimes \mathcal{O}_{\tilde{C}}(p-p')$. Since c belongs to P_0 and $\mathcal{O}_{\tilde{C}}(p-p')$ belongs to P_1, this pushdown is in $P_1 \subset \mathrm{Pic}^0(\tilde{C})$, the coset of the Prym variety which does not contain zero. Thus for c not in $a_p(\tilde{C})$, $f_*(\mathcal{L}_c)$ is a line bundle of degree zero on $\tilde{C}$ which is not trivial, hence has no sections. Thus $NR(\mathcal{E}_p) \subset a_p(\tilde{C})$. Since $\mathcal{L}_p = \mathcal{O}(\gamma^*(\mathcal{D}_p))$ is an effective standard Poincare bundle for every p in $\tilde{C}$, it follows that $NR(\mathcal{E}_p)$ contains a copy of $\tilde{C}$, so $NR(\mathcal{E}_p) = a_p(\tilde{C})$. In fact if $c = a_p(q)$, then $f_*(\mathcal{L}_c)) = \mathcal{O}_{\tilde{C}}(q)$ in $\mathrm{Pic}^1(\tilde{C})$, which is effective, with exactly one section. **QED.**

Again each line bundle $\mathcal{L}$ on $\tilde{C}\times\Xi$ induces by restriction two maps: $\Xi \to \mathrm{Pic}(\tilde{C})$ taking L to $\mathcal{L}|\tilde{C}\times\{L\}$, and $\tilde{C} \to \mathrm{Pic}(\Xi)$ taking q to $\mathcal{L}|\{q\}\times\Xi$. $\mathcal{L}$ is a Poincare bundle for $\tilde{C}\times\Xi$ if the first map $\Xi \to \mathrm{Pic}(\tilde{C})$ is the natural inclusion $\Xi \subset \mathrm{Pic}^{2g-2}(\tilde{C})$, and one can show that $c_1(\mu_*(\mathcal{L})) = \xi$ if and only if the second map $\tilde{C} \to \mathrm{Pic}(\Xi)$ goes into $\mathrm{Pic}^{\overline{\xi}}(\Xi)$.

The arguments above prove the following:

Corollary: If $\mathcal{E}$ is a sheaf as in (2) above, then $NR(\mathcal{E}) \cong$ {effective Poincare bundles $\mathcal{L}$ on $\tilde{C}\times\Xi$ with $c_1(\mu_*(\mathcal{L})) = \xi$} = {Poincare bundles of form $\mathcal{L}_p = \mathcal{O}(\gamma^*(\mathcal{D}_p))$ for p in $\tilde{C}$} = {Poincare bundles whose second restriction map is one of the natural maps $a_p:\tilde{C}\to \mathrm{Pic}^{\bar{\xi}}(\Xi)$ taking q to $\mathcal{O}_\Xi(\Xi_{a(q,p)})$}.

Relation of the factorization with the method of AMG.

There is also a connection between the factorization of the period maps through spaces of divisor varieties described above and the approach to Torelli of Andreotti Mayer and Green. The following discussion is to be taken more as suggestive than precise. We consider the Andreotti Mayer loci $\mathcal{N}_d = \{(A,\Theta): \dim(\mathrm{sing}(\Theta) \geq d\} \subset \mathcal{A}_g$. Then $\mathcal{J}(\mathcal{M}_g)\subset \mathcal{N}_{g-4}$ and $\mathcal{P}(\mathcal{R}_g)\subset\mathcal{N}_{p-6}$, where p = g-1 = dim(P). Consider the diagram of derivatives of the maps (considered as functors) from the diagrams above.

$$\begin{array}{ccc} & \alpha_* & \\ T_C\mathcal{M}_g & \rightarrow & T_C(g-1)\mathcal{C} \\ & \mathcal{J}_* \quad \wedge \beta_* & \\ & T_{\mathcal{J}(C)}\mathcal{N}_{g-4} & \end{array}$$

Then Torelli's theorem (for those curves C for which Enriques theorem applies) would follow from surjectivity of $\mathcal{J}_*$, as in 2.5 above. Kempf [Ke3] proved that α_* is surjective (when C is non hyperelliptic) and the approach to Green's theorem taken in 2.5, implies (at least when $\Theta(C)$

is irreducible) that β_* is surjective, completing the argument. The point is that if $\mathcal{J}_*$ is surjective, then there are not too many directions in which all singular points of Θ persist, hence there are relatively many equations, coming from quadric tangent cones to Θ, cutting out those directions, i.e. there are enough quadric tangent cones to Θ to define C_ω. Look at the analogous diagram for Pryms.

$$\begin{array}{ccc} T_{\tilde{C}/C}\mathcal{R}_g & \stackrel{\alpha_*}{\rightarrow} & T_X\mathcal{X} \\ & \mathcal{P}_* \quad \bigwedge \beta_* & \\ & T_{\mathcal{P}(\tilde{C}/C)}\mathcal{A}_{p-6} & \end{array}$$

By the "Enriques" theorems of [GL] and [LS], Prym Torelli would follow for $\tilde{C}/C$ with $\mathrm{Cliff}(C) \geq 3$, from surjectivity of $\mathcal{P}_*$. We have checked that injectivity of α_* holds for $g \geq 3$ and $\mathrm{Cliff}(C) \geq 1$. It remains to determine surjectivity of α_* and to analyze β_*. A key technical result in the analysis α_* for Pryms uses the other improper case of the equation in Lemma 4.5, i.e. the limit of $\varphi(\mathcal{D}_p \cap \mathcal{D}_q) = \Xi \cap \Xi_{a(q,p')}$ as $q \to p$. Then $h^0(\mathcal{O}_{\mathcal{D}_p}(\mathcal{D}_p)) = h^0(\mathcal{O}_\Xi(\Xi_{a(p,p')})) = 1$, implying that α_* is injective. Another application of the formula $h^0(\mathcal{O}_{\mathcal{D}_p}(\mathcal{D}_p)) = 1$, is to a proof that the Prym vector bundles $\mathcal{E}_p$ described above are "simple", i.e. that $\mathrm{End}_0(\mathcal{E}_p) \cong \mathbb{C}$. These results will appear elsewhere.

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